8

The Transportation and Assignment Problems

Chapter 3 emphasized the wide applicability of linear programming. We continue to broaden our horizons in this chapter by discussing two particularly important (and related) types of linear programming problems. One type, called the *transportation problem,* received this name because many of its applications involve determining how to optimally transport goods. However, some of its important applications (e.g., production scheduling) actually have nothing to do with transportation.

The second type, called the *assignment problem,* involves such applications as assigning people to tasks. Although its applications appear to be quite different from those for the transportation problem, we shall see that the assignment problem can be viewed as a special type of transportation problem.

The next chapter will introduce additional special types of linear programming problems involving *networks,* including the *minimum cost flow problem* (Sec. 9.6). There we shall see that both the transportation and assignment problems actually are special cases of the minimum cost flow problem. We introduce the network representation of the transportation and assignment problems in this chapter.

Applications of the transportation and assignment problems tend to require a very large number of constraints and variables, so a straightforward computer application of the simplex method may require an exorbitant computational effort. Fortunately, a key characteristic of these problems is that most of the a_{ij} coefficients in the constraints are zeros, and the relatively few nonzero coefficients appear in a distinctive pattern. As a result, it has been possible to develop special *streamlined* algorithms that achieve dramatic computational savings by exploiting this special structure of the problem. Therefore, it is important to become sufficiently familiar with these special types of problems that you can recognize them when they arise and apply the proper computational procedure.

To describe special structures, we shall introduce the table (matrix) of constraint coefficients shown in Table 8.1, where a_{ij} is the coefficient of the jth variable in the ith functional constraint. Later, portions of the table containing only coefficients equal to zero will be indicated by leaving them blank, whereas blocks containing nonzero coefficients will be shaded.

After presenting a prototype example for the transportation problem, we describe the special structure in its model and give additional examples of its applications. Section 8.2 presents the *transportation simplex method,* a special streamlined version of the simplex

TABLE 8.1 Table of constraint coefficients for linear programming

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

method for efficiently solving transportation problems. (You will see in Sec. 9.7 that this algorithm is related to the *network simplex method,* another streamlined version of the simplex method for efficiently solving any minimum cost flow problem, including both transportation and assignment problems.) Section 8.3 then focuses on the assignment problem.

OR TUTOR

8.1 THE TRANSPORTATION PROBLEM

Prototype Example

One of the main products of the P & T COMPANY is canned peas. The peas are prepared at three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and then shipped by truck to four distributing warehouses in the western United States (Sacramento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico), as shown in Fig. 8.1. Because the shipping costs are a major expense, management is initiating a study to reduce them as much as possible. For the upcoming season, an estimate has been made of the output from each cannery, and each warehouse has been allocated a certain amount from the total supply of peas. This information (in units of truckloads), along with the shipping cost per truckload for each cannery-warehouse combination, is given in Table 8.2. Thus, there are a total of 300 truckloads to be shipped. The problem now is to determine which plan for assigning these shipments to the various cannery-warehouse combinations would *minimize the total shipping cost.*

By ignoring the geographical layout of the canneries and warehouses, we can provide a *network representation* of this problem in a simple way by lining up all the canneries in one column on the left and all the warehouses in one column on the right. This representation is shown in Fig. 8.2. The arrows show the possible routes for the truckloads, where the number next to each arrow is the shipping cost per truckload for that route. A square bracket next to each location gives the number of truckloads to be shipped *out* of that location (so that the allocation into each warehouse is given as a negative number).

The problem depicted in Fig. 8.2 is actually a linear programming problem of the *transportation problem type*. To formulate the model, let Z denote total shipping cost, and let x_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3, 4$) be the number of truckloads to be shipped from cannery i to warehouse j. Thus, the objective is to choose the values of these 12 decision variables (the x_{ij}) so as to

$$\text{Minimize} \quad Z = 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},$$

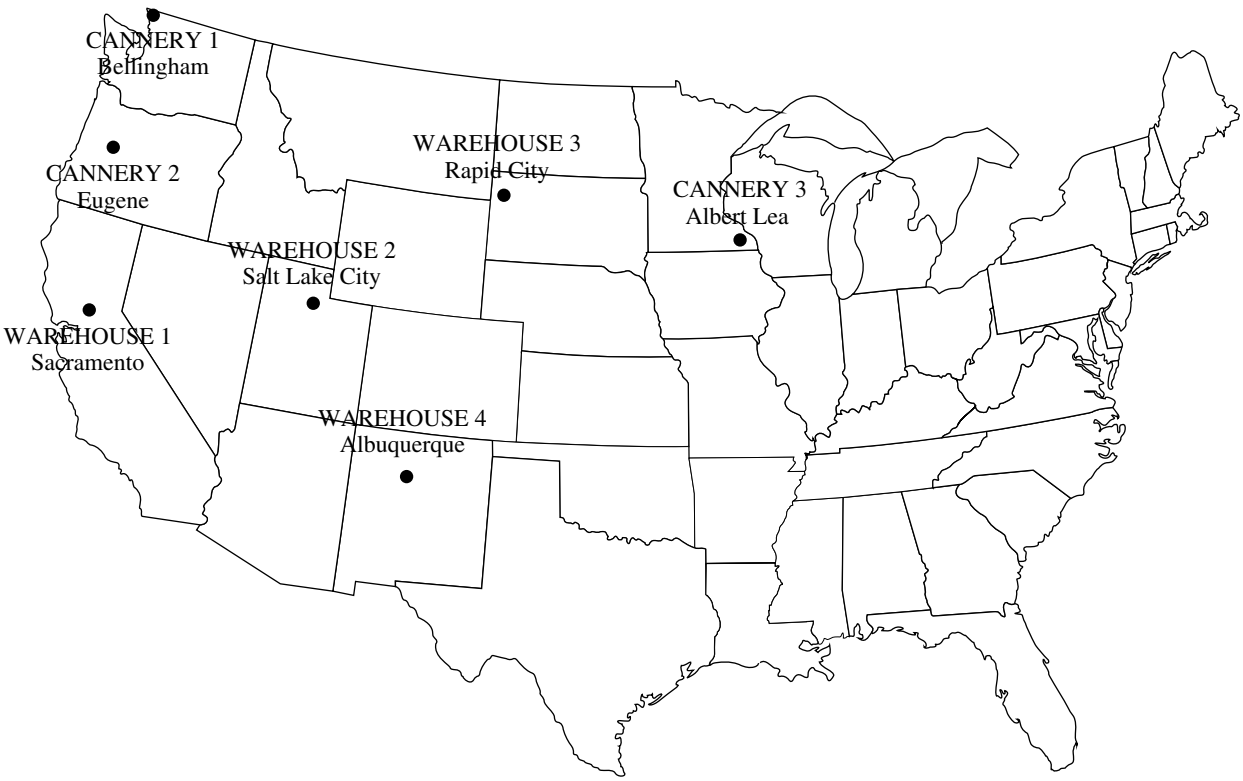


FIGURE 8.1
Location of canneries and warehouses for the P & T Co. problem.

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

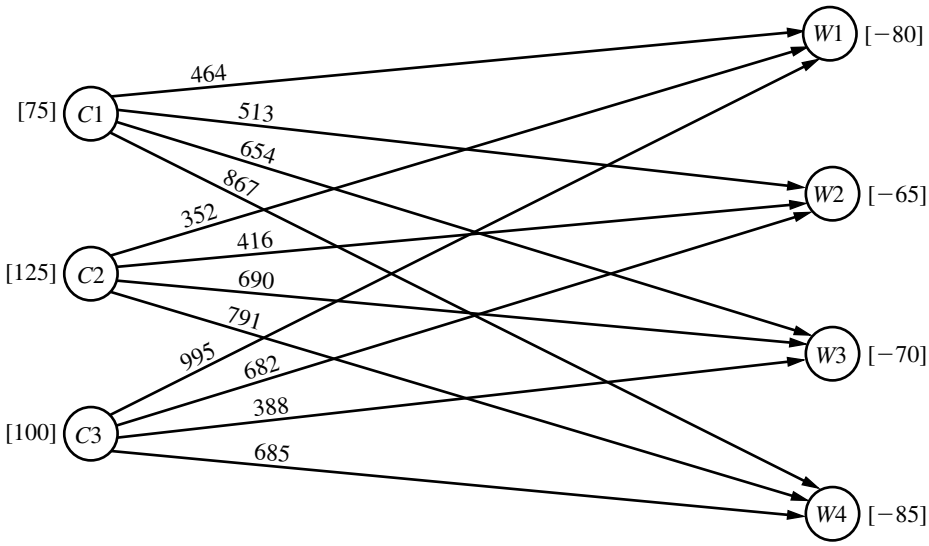


FIGURE 8.2
Network representation of the P & T Co. problem.

subject to the constraints

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
x_{11} + x_{21} + x_{31} &= 80 \\
x_{12} + x_{22} + x_{32} &= 65 \\
x_{13} + x_{23} + x_{33} &= 70 \\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

Table 8.3 shows the constraint coefficients. As you will see later in this section, it is the special structure in the pattern of these coefficients that distinguishes this problem as a transportation problem, not its context.

TABLE 8.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

However, before examining the special structure of the transportation problem model, let us pause to look at an actual application that resembles the P&T Co. problem but on a vastly larger scale.

An Award Winning Application of a Transportation Problem

Except for its small size, the P & T Co. problem is typical of the problems faced by many corporations which must ship goods from their manufacturing plants to their customers.

For example, consider an award winning OR study conducted at *Proctor & Gamble* (as described in the January–February 1997 issue of *Interfaces*). Prior to the study, the company's supply chain consisted of hundreds of suppliers, over 50 product categories, over 60 plants, 15 distribution centers, and over 1,000 customer zones. However, as the company moved toward global brands, management realized that it needed to consolidate plants to reduce manufacturing expenses, improve speed to market, and reduce capital investment. Therefore, the study focused on redesigning the company's production and distribution system for its North American operations. The result was a reduction in the number of North American plants by almost 20 percent, saving over $200 million in pretax costs per year.

A major part of the study revolved around formulating and solving transportation problems for individual product categories. For each option regarding the plants to keep open, etc., solving the corresponding transportation problem for a product category shows what the distribution cost would be for shipping the product category from those plants to the distribution centers and customer zones. Numerous such transportation problems were solved in the process of identifying the best new production and distribution system.

The Transportation Problem Model

To describe the general model for the transportation problem, we need to use terms that are considerably less specific than those for the components of the prototype example. In particular, the general transportation problem is concerned (literally or figuratively) with distributing *any* commodity from *any* group of supply centers, called **sources**, to *any* group of receiving centers, called **destinations,** in such a way as to minimize the total distribution cost. The correspondence in terminology between the prototype example and the general problem is summarized in Table 8.4.

As indicated by the fourth and fifth rows of the table, each source has a certain **supply** of units to distribute to the destinations, and each destination has a certain **demand**

TABLE 8.4 Terminology for the transportation problem

Prototype Example	General Problem
Truckloads of canned peas	Units of a commodity
Three canneries	m sources
Four warehouses	n destinations
Output from cannery i	Supply s_i from source i
Allocation to warehouse j	Demand d_j at destination j
Shipping cost per truckload from cannery i to warehouse j	Cost c_{ij} per unit distributed from source i to destination j

for units to be received from the sources. The model for a transportation problem makes the following assumption about these supplies and demands.

The requirements assumption: Each source has a fixed *supply* of units, where this entire supply must be distributed to the destinations. (We let s_i denote the number of units being supplied by source i, for $i = 1, 2, \ldots, m$.) Similarly, each destination has a fixed *demand* for units, where this entire demand must be received from the sources. (We let d_j denote the number of units being received by destination j, for $j = 1, 2, \ldots, n$.)

This assumption that there is no leeway in the amounts to be sent or received means that there needs to be a balance between the total supply from all sources and the total demand at all destinations.

The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Fortunately, these sums are equal for the P & T Co. since Table 8.2 indicates that the supplies (outputs) sum to 300 truckloads and so do the demands (allocations).

In some real problems, the supplies actually represent *maximum* amounts (rather than fixed amounts) to be distributed. Similarly, in other cases, the demands represent maximum amounts (rather than fixed amounts) to be received. Such problems do not quite fit the model for a transportation problem because they violate the *requirements assumption.* However, it is possible to *reformulate* the problem so that they then fit this model by introducing a *dummy destination* or a *dummy source* to take up the slack between the actual amounts and maximum amounts being distributed. We will illustrate how this is done with two examples at the end of this section.

The last row of Table 8.4 refers to a cost per unit distributed. This reference to a *unit cost* implies the following basic assumption for any transportation problem.

The cost assumption: The cost of distributing units from any particular source to any particular destination is *directly proportional* to the number of units distributed. Therefore, this cost is just the *unit cost* of distribution *times* the *number of units distributed.* (We let c_{ij} denote this unit cost for source i and destination j.)

The only data needed for a transportation problem model are the supplies, demands, and unit costs. These are the *parameters of the model.* All these parameters can be summarized conveniently in a single *parameter table* as shown in Table 8.5.

The model: Any problem (whether involving transportation or not) fits the model for a transportation problem if it can be described completely in terms of a *parameter table* like Table 8.5 and it satisfies both the *requirements assumption* and the *cost assumption.* The objective is to minimize the total cost of distributing the units. All the parameters of the model are included in this parameter table.

TABLE 8.5 Parameter table for the transportation problem

	Cost per Unit Distributed				
	Destination				
	1	**2**	⋯	***n***	**Supply**
Source 1	c_{11}	c_{12}	⋯	c_{1n}	s_1
2	c_{21}	c_{22}	⋯	c_{2n}	s_2
⋮	⋮	⋮	⋮	⋮	⋮
m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand	d_1	d_2	⋯	d_n	

Therefore, formulating a problem as a transportation problem only requires filling out a parameter table in the format of Table 8.5. Alternatively, the same information can be provided by using the network representation of the problem shown in Fig. 8.3. It is not necessary to write out a formal mathematical model.

However, we will go ahead and show you this model once for the general transportation problem just to emphasize that it is indeed a special type of linear programming problem.

Letting Z be the total distribution cost and x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the number of units to be distributed from source i to destination j, the linear programming formulation of this problem is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

Note that the resulting table of constraint coefficients has the special structure shown in Table 8.6. *Any* linear programming problem that fits this special formulation is of the transportation problem type, regardless of its physical context. In fact, there have been numerous applications unrelated to transportation that have been fitted to this special structure, as we shall illustrate in the next example later in this section. (The assignment problem described in Sec. 8.3 is an additional example.) This is one of the reasons why the transportation problem is considered such an important special type of linear programming problem.

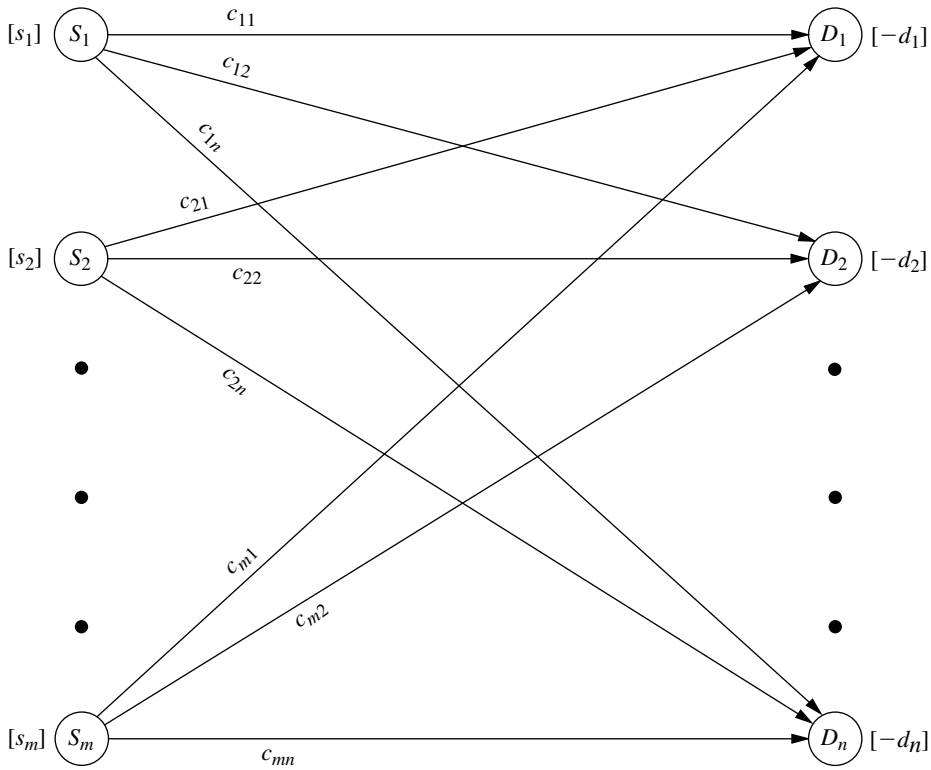


FIGURE 8.3
Network representation of the transportation problem.

MPL

MPL

MPL

MPL

MPL

MPL

For many applications, the supply and demand quantities in the model (the s_i and d_i) have integer values, and implementation will require that the distribution quantities (the x_{ij}) also have integer values. Fortunately, because of the special structure shown in Table 8.6, all such problems have the following property.

> **Integer solutions property:** For transportation problems where every s_i and d_j have an integer value, all the basic variables (allocations) in *every* basic feasible (BF) solution (including an optimal one) also have *integer* values.

The solution procedure described in Sec. 8.2 deals only with BF solutions, so it automatically will obtain an *integer* optimal solution for this case. (You will be able to see why this solution procedure actually gives a proof of the integer solutions property after you learn the procedure; Prob. 8.2-22 guides you through the reasoning involved.) Therefore, it is unnecessary to add a constraint to the model that the x_{ij} must have integer values.

As with other linear programming problems, the usual software options (Excel, LINGO/LINDO, MPL/CPLEX) are available to you for setting up and solving transportation problems (and assignment problems), as demonstrated in the files for this chapter in your OR Courseware. However, because the Excel approach now is somewhat different from what you have seen previously, we next describe this approach.

TABLE 8.6 Constraint coefficients for the transportation problem

	Coefficient of:											
	x_{11}	x_{12}	$\cdots$	x_{1n}	x_{21}	x_{22}	$\cdots$	x_{2n}	$\cdots$	x_{m1}	x_{m2}	$\cdots$	x_{mn}
A =	1	1	$\cdots$	1									
					1	1	$\cdots$	1					
									$\cdots$				
										1	1	$\cdots$	1
	1				1					1			
		1				1					1		
			$\cdots$				$\cdots$		$\cdots$			$\cdots$	
				1				1					1

Supply constraints

Demand constraints

Using Excel to Formulate and Solve Transportation Problems

To formulate and solve a transportation problem using Excel, two separate tables need to be entered on a spreadsheet. The first one is the parameter table. The second is a solution table, containing the quantities to distribute from each source to each destination. Figure 8.4 shows these two tables in rows 3–9 and 12–18 for the P&T Co. problem.

The two types of functional constraints need to be included in the spreadsheet. For the supply constraints, the total amount shipped from each source is calculated in column H of the solution table in Fig. 8.4. It is the sum of all the decision variable cells in the corresponding row. For example, the equation in cell H15 is "=D15+E15+F15+G15" or "=SUM(D15:G15)." The supply at each source is included in column J. Hence, the cells in column H must equal the corresponding cells in column J.

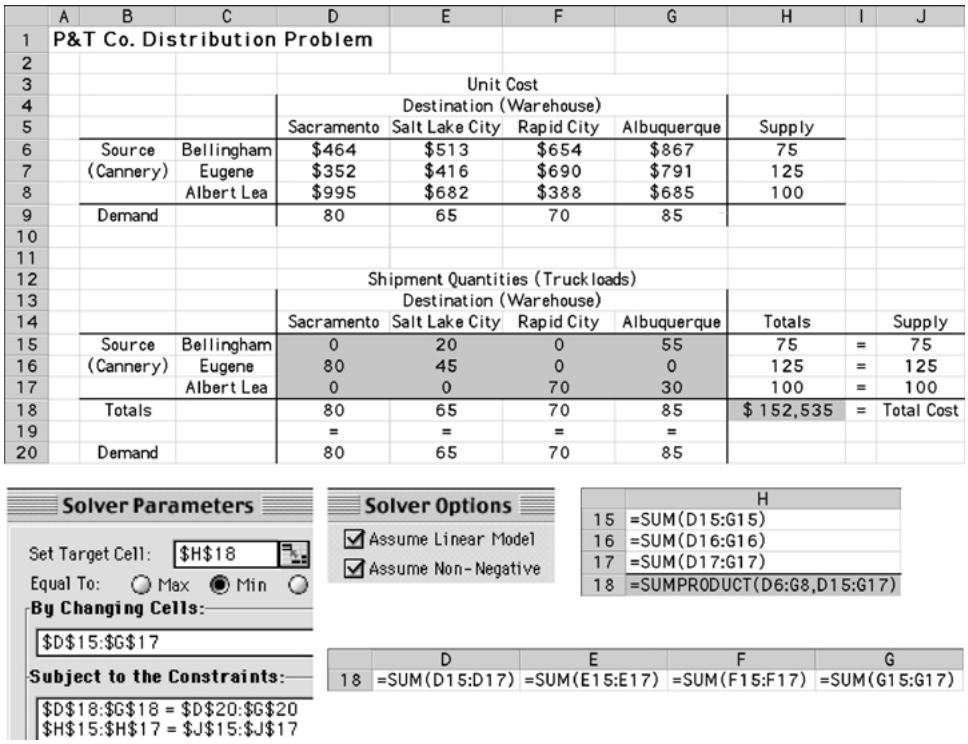
For the demand constraints, the total amount shipped to each destination is calculated in row 18 of the spreadsheet. For example, the equation in cell D18 is "=SUM(D15:D17)." The demand at each destination is then included in row 20.

The total cost is calculated in cell H18. This cost is the sum of the products of the corresponding cells in the main bodies of the parameter table and the solution table. Hence, the equation contained in cell H18 is "=SUMPRODUCT(D6:G8,D15:G17)."

Now let us look at the entries in the Solver dialogue box shown at the bottom of Fig. 8.4. These entries indicate that we are minimizing the total cost (calculated in cell H18) by changing the shipment quantities (in cells D15 through G17), subject to the constraints that the total amount shipped to each destination equals its demand (D18:G18=D20:G20) and that the total amount shipped from each source equals its supply (H15:H17=J15:J17). One of the selected Solver options (Assume Non-Negative) specifies that all shipment quantities must be nonnegative. The other one (Assume Linear Model) indicates that this transportation problem is a linear programming problem.

The values of the x_{ij} decision variables (the shipment quantities) are contained in the changing cells (D15:G17). To begin, any value (such as 0) can be entered in each of these cells. After clicking on the Solve button, the Solver will use the simplex method to solve the problem. The optimal solution obtained in this way is shown in the changing cells in Fig. 8.4, along with the resulting total cost in cell H18.

Note that the Solver simply uses the general simplex method to solve a transportation problem rather than a streamlined version that is specially designed for solving trans-



	A	B	C	D	E	F	G	H	I	J
1	P&T Co. Distribution Problem									
2										
3				Unit Cost						
4				Destination (Warehouse)						
5				Sacramento	Salt Lake City	Rapid City	Albuquerque	Supply		
6		Source	Bellingham	$464	$513	$654	$867	75		
7		(Cannery)	Eugene	$352	$416	$690	$791	125		
8			Albert Lea	$995	$682	$388	$685	100		
9		Demand		80	65	70	85			
10										
11										
12				Shipment Quantities (Truckloads)						
13				Destination (Warehouse)						
14				Sacramento	Salt Lake City	Rapid City	Albuquerque	Totals		Supply
15		Source	Bellingham	0	20	0	55	75	=	75
16		(Cannery)	Eugene	80	45	0	0	125	=	125
17			Albert Lea	0	0	70	30	100	=	100
18		Totals		80	65	70	85	$ 152,535	=	Total Cost
19				=	=	=	=			
20		Demand		80	65	70	85			

	H
15	=SUM(D15:G15)
16	=SUM(D16:G16)
17	=SUM(D17:G17)
18	=SUMPRODUCT(D6:G8,D15:G17)

	D	E	F	G
18	=SUM(D15:D17)	=SUM(E15:E17)	=SUM(F15:F17)	=SUM(G15:G17)

FIGURE 8.4
A spreadsheet formulation of the P & T Co. problem as a transportation problem, where rows 3 to 9 show the parameter table and rows 12 to 18 display the solution table after using the Excel Solver to obtain an optimal shipping plan. Both the formulas for the output cells and the specifications needed to set up the Solver are given at the bottom.

portation problems very efficiently, such as the transportation simplex method presented in the next section. Therefore, a software package that includes such a streamlined version should solve a large transportation problem much faster than the Excel Solver.

We mentioned earlier that some problems do not quite fit the model for a transportation problem because they violate the requirements assumption, but that it is possible to reformulate such a problem to fit this model by introducing a dummy destination or a dummy source. When using the Excel Solver, it is not necessary to do this reformulation since the simplex method can solve the original model where the supply constraints are in $\leq$ form or the demand constraints are in $\geq$ form. However, the larger the problem, the more worthwhile it becomes to do the reformulation and use the transportation simplex method (or equivalent) instead with another software package.

The next two examples illustrate how to do this kind of reformulation.

An Example with a Dummy Destination

The NORTHERN AIRPLANE COMPANY builds commercial airplanes for various airline companies around the world. The last stage in the production process is to produce the jet engines and then to install them (a very fast operation) in the completed airplane frame. The company has been working under some contracts to deliver a considerable number of airplanes in the near future, and the production of the jet engines for these planes must now be scheduled for the next 4 months.

TABLE 8.7 Production scheduling data for Northern Airplane Co.

Month	Scheduled Installations	Maximum Production	Unit Cost* of Production	Unit Cost* of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

*Cost is expressed in millions of dollars.

To meet the contracted dates for delivery, the company must supply engines for installation in the quantities indicated in the second column of Table 8.7. Thus, the cumulative number of engines produced by the end of months 1, 2, 3, and 4 must be at least 10, 25, 50, and 70, respectively.

The facilities that will be available for producing the engines vary according to other production, maintenance, and renovation work scheduled during this period. The resulting monthly differences in the maximum number that can be produced and the cost (in millions of dollars) of producing each one are given in the third and fourth columns of Table 8.7.

Because of the variations in production costs, it may well be worthwhile to produce some of the engines a month or more before they are scheduled for installation, and this possibility is being considered. The drawback is that such engines must be stored until the scheduled installation (the airplane frames will not be ready early) at a storage cost of \$15,000 per month (including interest on expended capital) for each engine,[1] as shown in the rightmost column of Table 8.7.

The production manager wants a schedule developed for the number of engines to be produced in each of the 4 months so that the total of the production and storage costs will be minimized.

Formulation. One way to formulate a mathematical model for this problem is to let x_j be the number of jet engines to be produced in month j, for $j = 1, 2, 3, 4$. By using only these four decision variables, the problem can be formulated as a linear programming problem that does *not* fit the transportation problem type. (See Prob. 8.2-20.)

On the other hand, by adopting a different viewpoint, we can instead formulate the problem as a transportation problem that requires *much* less effort to solve. This viewpoint will describe the problem in terms of sources and destinations and then identify the corresponding x_{ij}, c_{ij}, s_i, and d_j. (See if you can do this before reading further.)

Because the units being distributed are jet engines, each of which is to be scheduled for production in a particular month and then installed in a particular (perhaps different) month,

Source i = production of jet engines in month i ($i = 1, 2, 3, 4$)

Destination j = installation of jet engines in month j ($j = 1, 2, 3, 4$)

[1]For modeling purposes, assume that this storage cost is incurred at the *end of the month* for just those engines that are being held over into the next month. Thus, engines that are produced in a given month for installation in the same month are assumed to incur no storage cost.

x_{ij} = number of engines produced in month i for installation in month j

c_{ij} = cost associated with each unit of x_{ij}

$$= \begin{cases} \text{cost per unit for production and any storage} & \text{if } i \leq j \\ ? & \text{if } i > j \end{cases}$$

$s_i = ?$

d_j = number of scheduled installations in month j.

The corresponding (incomplete) parameter table is given in Table 8.8. Thus, it remains to identify the missing costs and the supplies.

Since it is impossible to produce engines in one month for installation in an earlier month, x_{ij} must be zero if $i > j$. Therefore, there is no real cost that can be associated with such x_{ij}. Nevertheless, in order to have a well-defined transportation problem to which the solution procedure of Sec. 8.2 can be applied, it is necessary to assign some value for the unidentified costs. Fortunately, we can use the *Big M method* introduced in Sec. 4.6 to assign this value. Thus, we assign a *very* large number (denoted by M for convenience) to the unidentified cost entries in Table 8.8 to force the corresponding values of x_{ij} to be zero in the final solution.

The numbers that need to be inserted into the supply column of Table 8.8 are not obvious because the "supplies," the amounts produced in the respective months, are not fixed quantities. In fact, the objective is to solve for the most desirable values of these production quantities. Nevertheless, it is necessary to assign some fixed number to every entry in the table, including those in the supply column, to have a transportation problem. A clue is provided by the fact that although the supply constraints are not present in the usual form, these constraints do exist in the form of upper bounds on the amount that can be supplied, namely,

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 25,$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 35,$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 30,$$
$$x_{41} + x_{42} + x_{43} + x_{44} \leq 10.$$

The only change from the standard model for the transportation problem is that these constraints are in the form of inequalities instead of equalities.

TABLE 8.8 Incomplete parameter table for Northern Airplane Co.

		Cost per Unit Distributed				
		Destination				
		1	2	3	4	Supply
Source	1	1.080	1.095	1.110	1.125	?
	2	?	1.110	1.125	1.140	?
	3	?	?	1.100	1.115	?
	4	?	?	?	1.130	?
Demand		10	15	25	20	

TABLE 8.9 Complete parameter table for Northern Airplane Co.

		Cost per Unit Distributed					
		Destination					
		1	2	3	4	5(*D*)	Supply
Source	1	1.080	1.095	1.110	1.125	0	25
	2	*M*	1.110	1.125	1.140	0	35
	3	*M*	*M*	1.100	1.115	0	30
	4	*M*	*M*	*M*	1.130	0	10
Demand		10	15	25	20	30	

To convert these inequalities to equations in order to fit the transportation problem model, we use the familiar device of *slack variables,* introduced in Sec. 4.2. In this context, the slack variables are allocations to a single **dummy destination** that represent the *unused production capacity* in the respective months. This change permits the supply in the transportation problem formulation to be the total production capacity in the given month. Furthermore, because the demand for the dummy destination is the total unused capacity, this demand is

$$(25 + 35 + 30 + 10) - (10 + 15 + 25 + 20) = 30.$$

With this demand included, the sum of the supplies now equals the sum of the demands, which is the condition given by the *feasible solutions property* for having feasible solutions.

The cost entries associated with the dummy destination should be zero because there is no cost incurred by a fictional allocation. (Cost entries of M would be *inappropriate* for this column because we do not want to force the corresponding values of x_{ij} to be zero. In fact, these values need to sum to 30.)

The resulting final parameter table is given in Table 8.9, with the dummy destination labeled as destination 5(D). By using this formulation, it is quite easy to find the optimal production schedule by the solution procedure described in Sec. 8.2. (See Prob. 8.2-11 and its answer in the back of the book.)

An Example with a Dummy Source

METRO WATER DISTRICT is an agency that administers water distribution in a large geographic region. The region is fairly arid, so the district must purchase and bring in water from outside the region. The sources of this imported water are the Colombo, Sacron, and Calorie rivers. The district then resells the water to users in the region. Its main customers are the water departments of the cities of Berdoo, Los Devils, San Go, and Hollyglass.

It is possible to supply any of these cities with water brought in from any of the three rivers, with the exception that no provision has been made to supply Hollyglass with Calorie River water. However, because of the geographic layouts of the aqueducts and the cities in the region, the cost to the district of supplying water depends upon both the source of

TABLE 8.10 Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				
	Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	—	50
Minimum needed	30	70	0	10	(in units of 1
Requested	50	70	30	∞	million acre feet)

the water and the city being supplied. The variable cost per acre foot of water (in tens of dollars) for each combination of river and city is given in Table 8.10. Despite these variations, the price per acre foot charged by the district is independent of the source of the water and is the same for all cities.

The management of the district is now faced with the problem of how to allocate the available water during the upcoming summer season. In units of 1 million acre feet, the amounts available from the three rivers are given in the rightmost column of Table 8.10. The district is committed to providing a certain minimum amount to meet the essential needs of each city (with the exception of San Go, which has an independent source of water), as shown in the *minimum needed* row of the table. The *requested* row indicates that Los Devils desires no more than the minimum amount, but that Berdoo would like to buy as much as 20 more, San Go would buy up to 30 more, and Hollyglass will take as much as it can get.

Management wishes to allocate *all* the available water from the three rivers to the four cities in such a way as to at least meet the essential needs of each city while minimizing the total cost to the district.

Formulation. Table 8.10 already is close to the proper form for a parameter table, with the rivers being the sources and the cities being the destinations. However, the one basic difficulty is that it is not clear what the demands at the destinations should be. The amount to be received at each destination (except Los Devils) actually is a decision variable, with both a lower bound and an upper bound. This upper bound is the amount requested unless the request exceeds the total supply remaining after the minimum needs of the other cities are met, in which case this *remaining supply* becomes the upper bound. Thus, insatiably thirsty Hollyglass has an upper bound of

$$(50 + 60 + 50) - (30 + 70 + 0) = 60.$$

Unfortunately, just like the other numbers in the parameter table of a transportation problem, the demand quantities must be *constants,* not bounded decision variables. To begin resolving this difficulty, temporarily suppose that it is not necessary to satisfy the minimum needs, so that the upper bounds are the only constraints on amounts to be allocated to the cities. In this circumstance, can the requested allocations be viewed as the demand quantities for a transportation problem formulation? After one adjustment, yes! (Do you see already what the needed adjustment is?)

The situation is analogous to Northern Airplane Co.'s production scheduling problem, where there was *excess supply capacity.* Now there is *excess demand capacity.* Consequently, rather than introducing a *dummy destination* to "receive" the unused supply capacity, the adjustment needed here is to introduce a **dummy source** to "send" the *unused demand capacity.* The imaginary supply quantity for this dummy source would be the amount by which the sum of the demands exceeds the sum of the real supplies:

$$(50 + 70 + 30 + 60) - (50 + 60 + 50) = 50.$$

This formulation yields the parameter table shown in Table 8.11, which uses units of million acre feet and tens of millions of dollars. The cost entries in the *dummy* row are zero because there is no cost incurred by the fictional allocations from this dummy source. On the other hand, a huge unit cost of M is assigned to the Calorie River–Hollyglass spot. The reason is that Calorie River water cannot be used to supply Hollyglass, and assigning a cost of M will prevent any such allocation.

Now let us see how we can take each city's minimum needs into account in this kind of formulation. Because San Go has no minimum need, it is all set. Similarly, the formulation for Hollyglass does not require any adjustments because its demand (60) exceeds the dummy source's supply (50) by 10, so the amount supplied to Hollyglass from the *real* sources will be *at least 10* in any feasible solution. Consequently, its minimum need of 10 from the rivers is guaranteed. (If this coincidence had not occurred, Hollyglass would need the same adjustments that we shall have to make for Berdoo.)

Los Devils' minimum need equals its requested allocation, so its *entire* demand of 70 must be filled from the real sources rather than the dummy source. This requirement calls for the Big M method! Assigning a huge unit cost of M to the allocation from the dummy source to Los Devils ensures that this allocation will be zero in an optimal solution.

Finally, consider Berdoo. In contrast to Hollyglass, the dummy source has an adequate (fictional) supply to "provide" at least some of Berdoo's minimum need in addition to its extra requested amount. Therefore, since Berdoo's minimum need is 30, adjustments must be made to prevent the dummy source from contributing more than 20 to Berdoo's total demand of 50. This adjustment is accomplished by splitting Berdoo into two destinations, one having a demand of 30 with a unit cost of M for any allocation from the dummy source and the other having a demand of 20 with a unit cost of zero for the dummy source allocation. This formulation gives the final parameter table shown in Table 8.12.

TABLE 8.11 Parameter table without minimum needs for Metro Water District

		Cost (Tens of Millions of Dollars) per Unit Distributed				
		Destination				
		Berdoo	Los Devils	San Go	Hollyglass	Supply
Source	Colombo River	16	13	22	17	50
	Sacron River	14	13	19	15	60
	Calorie River	19	20	23	*M*	50
	Dummy	0	0	0	0	50
Demand		50	70	30	60	

TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	*M*	50
	Dummy	4(*D*)	*M*	0	*M*	0	0	50
Demand			30	20	70	30	60	

This problem will be solved in the next section to illustrate the solution procedure presented there.

8.2 A STREAMLINED SIMPLEX METHOD FOR THE TRANSPORTATION PROBLEM

Because the transportation problem is just a special type of linear programming problem, it can be solved by applying the simplex method as described in Chap. 4. However, you will see in this section that some tremendous computational shortcuts can be taken in this method by exploiting the special structure shown in Table 8.6. We shall refer to this streamlined procedure as the **transportation simplex method.**

As you read on, note particularly how the special structure is exploited to achieve great computational savings. This will illustrate an important OR technique—streamlining an algorithm to exploit the special structure in the problem at hand.

Setting Up the Transportation Simplex Method

To highlight the streamlining achieved by the transportation simplex method, let us first review how the general (unstreamlined) simplex method would set up a transportation problem in tabular form. After constructing the table of constraint coefficients (see Table 8.6), converting the objective function to maximization form, and using the Big *M* method to introduce artificial variables $z_1, z_2, \ldots, z_{m+n}$ into the $m + n$ respective equality constraints (see Sec. 4.6), typical columns of the simplex tableau would have the form shown in Table 8.13, where all entries *not shown* in these columns are *zeros.* [The one remaining adjustment to be made before the first iteration of the simplex method is to algebraically eliminate the nonzero coefficients of the initial (artificial) basic variables in row 0.]

After any subsequent iteration, row 0 then would have the form shown in Table 8.14. Because of the pattern of 0s and 1s for the coefficients in Table 8.13, by the *fundamental insight* presented in Sec. 5.3, u_i and v_j would have the following interpretation:

u_i = multiple of *original* row i that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

TABLE 8.13 Original simplex tableau before simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of:								Right side
		Z	⋯	x_{ij}	⋯	z_i	⋯	z_{m+j}	⋯	
Z	(0)	-1		c_{ij}		M		M		0
	(1)									
	⋮									
z_i	(i)	0		1		1				s_i
	⋮									
z_{m+j}	($m + j$)	0		1				1		d_j
	⋮									
	($m + n$)									

v_j = multiple of *original* row $m + j$ that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

Using the duality theory introduced in Chap. 6, another property of the u_i and v_j is that they are the *dual variables.*[1] If x_{ij} is a nonbasic variable, $c_{ij} - u_i - v_j$ is interpreted as the rate at which Z will change as x_{ij} is increased.

To lay the groundwork for simplifying this setup, recall what information is needed by the simplex method. In the initialization, an initial BF solution must be obtained, which is done artificially by introducing artificial variables as the initial basic variables and setting them equal to s_i and d_j. The optimality test and step 1 of an iteration (selecting an entering basic variable) require knowing the current row 0, which is obtained by subtracting a certain multiple of another row from the preceding row 0. Step 2 (determining the leaving basic variable) must identify the basic variable that reaches zero first as the entering basic variable is increased, which is done by comparing the current coefficients of the entering basic variable and the corresponding right side. Step 3 must determine the new BF solution, which is found by subtracting certain multiples of one row from the other rows in the current simplex tableau.

Now, how does the *transportation simplex method* obtain the same information in much simpler ways? This story will unfold fully in the coming pages, but here are some preliminary answers.

First, *no artificial variables* are needed, because a simple and convenient procedure (with several variations) is available for constructing an initial BF solution.

Second, the current row 0 can be obtained *without using any other row* simply by calculating the current values of u_i and v_j directly. Since each basic variable must have a coefficient of zero in row 0, the current u_i and v_j are obtained by solving the set of equations

$$c_{ij} - u_i - v_j = 0 \qquad \text{for each } i \text{ and } j \text{ such that } x_{ij} \text{ is a basic variable.}$$

[1]It would be easier to recognize these variables as dual variables by relabeling all these variables as y_i and then changing all the signs in row 0 of Table 8.14 by converting the objective function back to its original minimization form.

TABLE 8.14 Row 0 of simplex tableau when simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of:								Right Side
		Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	
Z	(0)	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

(We will illustrate this straightforward procedure later when discussing the optimality test for the transportation simplex method.) The special structure in Table 8.13 makes this convenient way of obtaining row 0 possible by yielding $c_{ij} - u_i - v_j$ as the coefficient of x_{ij} in Table 8.14.

Third, the leaving basic variable can be identified in a simple way without (explicitly) using the coefficients of the entering basic variable. The reason is that the special structure of the problem makes it easy to see how the solution must change as the entering basic variable is increased. As a result, the new BF solution also can be identified immediately *without any algebraic manipulations* on the rows of the simplex tableau. (You will see the details when we describe how the transportation simplex method performs an iteration.)

The grand conclusion is that *almost the entire simplex tableau* (and the work of maintaining it) *can be eliminated*! Besides the input data (the c_{ij}, s_i, and d_j values), the only information needed by the transportation simplex method is the current BF solution,[1] the current values of u_i and v_j, and the resulting values of $c_{ij} - u_i - v_j$ for nonbasic variables x_{ij}. When you solve a problem by hand, it is convenient to record this information for each iteration in a **transportation simplex tableau**, such as shown in Table 8.15. (Note carefully that the values of x_{ij} and $c_{ij} - u_i - v_j$ are distinguished in these tableaux by circling the former but not the latter.)

You can gain a fuller appreciation for the great difference in efficiency and convenience between the simplex and the transportation simplex methods by applying both to the same small problem (see Prob. 8.2-19). However, the difference becomes even more pronounced for large problems that must be solved on a computer. This pronounced difference is suggested somewhat by comparing the sizes of the simplex and the transportation simplex tableaux. Thus, for a transportation problem having m sources and n destinations, the simplex tableau would have $m + n + 1$ rows and $(m + 1)(n + 1)$ columns (excluding those to the left of the x_{ij} columns), and the transportation simplex tableau would have m rows and n columns (excluding the two extra informational rows and columns). Now try plugging in various values for m and n (for example, $m = 10$ and $n = 100$ would be a rather typical medium-size transportation problem), and note how the ratio of the number of cells in the simplex tableau to the number in the transportation simplex tableau increases as m and n increase.

[1]Since nonbasic variables are automatically zero, the current BF solution is fully identified by recording just the values of the basic variables. We shall use this convention from now on.

TABLE 8.15 Format of a transportation simplex tableau

		Destination					
		1	**2**	⋯	***n***	**Supply**	u_i
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1	
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2	
	⋮	⋯	⋯	⋯	⋯	⋮	
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m	
Demand		d_1	d_2	⋯	d_n	$Z =$	
	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable	*If x_{ij} is a nonbasic variable*
c_{ij} (x_{ij})	c_{ij} $c_{ij} - u_i - v_j$

Initialization

Recall that the objective of the initialization is to obtain an initial BF solution. Because all the functional constraints in the transportation problem are *equality* constraints, the simplex method would obtain this solution by introducing artificial variables and using them as the initial basic variables, as described in Sec. 4.6. The resulting basic solution actually is feasible only for a revised version of the problem, so a number of iterations are needed to drive these artificial variables to zero in order to reach the real BF solutions. The transportation simplex method bypasses all this by instead using a simpler procedure to directly construct a real BF solution on a transportation simplex tableau.

Before outlining this procedure, we need to point out that the number of basic variables in any basic solution of a transportation problem is one fewer than you might expect. Ordinarily, there is one basic variable for each functional constraint in a linear programming problem. For transportation problems with m sources and n destinations, the number of functional constraints is $m + n$. However,

Number of basic variables $= m + n - 1$.

The reason is that the functional constraints are equality constraints, and this set of $m + n$ equations has one *extra* (or *redundant*) equation that can be deleted without changing the feasible region; i.e., any one of the constraints is automatically satisfied whenever the other $m + n - 1$ constraints are satisfied. (This fact can be verified by showing that any supply constraint exactly equals the sum of the demand constraints minus the sum of the *other* supply constraints, and that any demand equation also can be reproduced by summing the supply equations and subtracting the other demand equations. See Prob.

8.2-21.) Therefore, any *BF solution* appears on a transportation simplex tableau with exactly $m + n - 1$ circled *nonnegative* allocations, where the sum of the allocations for each row or column equals its supply or demand.[1]

The procedure for constructing an initial BF solution selects the $m + n - 1$ basic variables one at a time. After each selection, a value that will satisfy one additional constraint (thereby eliminating that constraint's row or column from further consideration for providing allocations) is assigned to that variable. Thus, after $m + n - 1$ selections, an entire basic solution has been constructed in such a way as to satisfy all the constraints. A number of different criteria have been proposed for selecting the basic variables. We present and illustrate three of these criteria here, after outlining the general procedure.

General Procedure[2] for Constructing an Initial BF Solution. To begin, all source rows and destination columns of the transportation simplex tableau are initially under consideration for providing a basic variable (allocation).

1. From the rows and columns still under consideration, select the next basic variable (allocation) according to some criterion.
2. Make that allocation large enough to exactly use up the remaining supply in its row or the remaining demand in its column (whichever is smaller).
3. Eliminate that row or column (whichever had the smaller remaining supply or demand) from further consideration. (If the row and column have the same remaining supply and demand, then arbitrarily select the *row* as the one to be eliminated. The column will be used later to provide a *degenerate* basic variable, i.e., a circled allocation of zero.)
4. If only one row or only one column remains under consideration, then the procedure is completed by selecting every *remaining* variable (i.e., those variables that were neither previously selected to be basic nor eliminated from consideration by eliminating their row or column) associated with that row or column to be basic with the only feasible allocation. Otherwise, return to step 1.

Alternative Criteria for Step 1

1. *Northwest corner rule:* Begin by selecting x_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ (that is, move one column to the *right*) if source i has any supply remaining. Otherwise, next select $x_{i+1,j}$ (that is, move one row *down*).

Example. To make this description more concrete, we now illustrate the general procedure on the Metro Water District problem (see Table 8.12) with the northwest corner rule being used in step 1. Because $m = 4$ and $n = 5$ in this case, the procedure would find an initial BF solution having $m + n - 1 = 8$ basic variables.

[1]However, note that any feasible solution with $m + n - 1$ nonzero variables is *not necessarily* a basic solution because it might be the weighted average of two or more degenerate BF solutions (i.e., BF solutions having some basic variables equal to zero). We need not be concerned about mislabeling such solutions as being basic, however, because the transportation simplex method constructs only legitimate BF solutions.

[2]In Sec. 4.1 we pointed out that the simplex method is an example of the algorithms (systematic solution procedures) so prevalent in OR work. Note that this procedure also is an algorithm, where each successive execution of the (four) steps constitutes an iteration.

TABLE 8.16 Initial BF solution from the Northwest Corner Rule

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 (30) →	16 (20) ↓	13	22	17	50	
	2	14	14 (0) →	13 (60) ↓	19	15	60	
	3	19	19	20 (10) →	23 (30) →	*M* (10) ↓	50	
	4(*D*)	*M*	0	*M*	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

As shown in Table 8.16, the first allocation is $x_{11} = 30$, which exactly uses up the demand in column 1 (and eliminates this column from further consideration). This first iteration leaves a supply of 20 remaining in row 1, so next select $x_{1,1+1} = x_{12}$ to be a basic variable. Because this supply is no larger than the demand of 20 in column 2, all of it is allocated, $x_{12} = 20$, and this row is eliminated from further consideration. (Row 1 is chosen for elimination rather than column 2 because of the parenthetical instruction in step 3.) Therefore, select $x_{1+1,2} = x_{22}$ next. Because the remaining demand of 0 in column 2 is less than the supply of 60 in row 2, allocate $x_{22} = 0$ and eliminate column 2.

Continuing in this manner, we eventually obtain the entire *initial BF solution* shown in Table 8.16, where the circled numbers are the values of the basic variables ($x_{11} = 30$, . . . , $x_{45} = 50$) and all the other variables (x_{13}, etc.) are nonbasic variables equal to zero. Arrows have been added to show the order in which the basic variables (allocations) were selected. The value of Z for this solution is

$$Z = 16(30) + 16(20) + \cdots + 0(50) = 2{,}470 + 10M.$$

2. *Vogel's approximation method:* For each row and column remaining under consideration, calculate its **difference,** which is defined as *the arithmetic difference between the smallest and next-to-the-smallest unit cost c_{ij} still remaining in that row or column.* (If two unit costs tie for being the smallest remaining in a row or column, then the *difference* is 0.) In that row or column having the *largest difference,* select the variable having the *smallest remaining unit cost.* (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

Example. Now let us apply the general procedure to the Metro Water District problem by using the criterion for Vogel's approximation method to select the next basic variable in step 1. With this criterion, it is more convenient to work with parameter tables (rather

than with complete transportation simplex tableaux), beginning with the one shown in Table 8.12. At each iteration, after the difference for every row and column remaining under consideration is calculated and displayed, the largest difference is circled and the smallest unit cost in its row or column is enclosed in a box. The resulting selection (and value) of the variable having this unit cost as the next basic variable is indicated in the lower right-hand corner of the current table, along with the row or column thereby being eliminated from further consideration (see steps 2 and 3 of the general procedure). The table for the next iteration is exactly the same except for deleting this row or column and subtracting the last allocation from its supply or demand (whichever remains).

Applying this procedure to the Metro Water District problem yields the sequence of parameter tables shown in Table 8.17, where the resulting initial BF solution consists of the eight basic variables (allocations) given in the lower right-hand corner of the respective parameter tables.

This example illustrates two relatively subtle features of the general procedure that warrant special attention. First, note that the final iteration selects *three* variables (x_{31}, x_{32}, and x_{33}) to become basic instead of the single selection made at the other iterations. The reason is that only *one* row (row 3) remains under consideration at this point. Therefore, step 4 of the general procedure says to select *every* remaining variable associated with row 3 to be basic.

Second, note that the allocation of $x_{23} = 20$ at the next-to-last iteration exhausts *both* the remaining supply in its row *and* the remaining demand in its column. However, rather than eliminate both the row and column from further consideration, step 3 says to eliminate *only the row,* saving the column to provide a *degenerate* basic variable later. Column 3 is, in fact, used for just this purpose at the final iteration when $x_{33} = 0$ is selected as one of the basic variables. For another illustration of this same phenomenon, see Table 8.16 where the allocation of $x_{12} = 20$ results in eliminating only row 1, so that column 2 is saved to provide a degenerate basic variable, $x_{22} = 0$, at the next iteration.

Although a zero allocation might seem irrelevant, it actually plays an important role. You will see soon that the transportation simplex method must know *all* $m + n - 1$ basic variables, including those with value zero, in the current BF solution.

3. *Russell's approximation method:* For each source row i remaining under consideration, determine its $\overline{u}_i$, which is the largest unit cost c_{ij} still remaining in that row. For each destination column j remaining under consideration, determine its $\overline{v}_j$, which is the largest unit cost c_{ij} still remaining in that column. For each variable x_{ij} not previously selected in these rows and columns, calculate $\Delta_{ij} = c_{ij} - \overline{u}_i - \overline{v}_j$. Select the variable having the *largest* (in absolute terms) *negative* value of Δ_{ij}. (Ties may be broken arbitrarily.)

Example. Using the criterion for Russell's approximation method in step 1, we again apply the general procedure to the Metro Water District problem (see Table 8.12). The results, including the sequence of basic variables (allocations), are shown in Table 8.18.

At iteration 1, the largest unit cost in row 1 is $\overline{u}_1 = 22$, the largest in column 1 is $\overline{v}_1 = M$, and so forth. Thus,

$$\Delta_{11} = c_{11} - \overline{u}_1 - \overline{v}_1 = 16 - 22 - M = -6 - M.$$

TABLE 8.17 Initial BF solution from Vogel's approximation method

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	*M*	50	0
	4(*D*)	*M*	0	*M*	[0]	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	(19)	15	Eliminate column 4	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
	4(*D*)	*M*	0	*M*	[0]	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	(15)	Eliminate row 4(*D*)	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	[13]	17	50	(3)
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	2	14	14	13	[15]	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	(*M* − 15)	Eliminate column 5	

		Destination				
		1	**2**	**3**	**Supply**	**Row Difference**
Source	2	14	14	[13]	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	(7)	Eliminate row 2	

		Destination				
		1	**2**	**3**	**Supply**	
Source	3	19	19	20	50	
Demand		30	20	0	Select $x_{31} = 30$ $x_{32} = 20$ $x_{33} = 0$	$Z = 2{,}460$

TABLE 8.18 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

Calculating all the Δ_{ij} values for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$ shows that $\Delta_{45} = 0 - 2M$ has the largest negative value, so $x_{45} = 50$ is selected as the first basic variable (allocation). This allocation exactly uses up the supply in row 4, so this row is eliminated from further consideration.

Note that eliminating this row changes $\bar{v}_1$ and $\bar{v}_3$ for the next iteration. Therefore, the second iteration requires recalculating the Δ_{ij} with $j = 1, 3$ as well as eliminating $i = 4$. The largest negative value now is

$$\Delta_{15} = 17 - 22 - M = -5 - M,$$

so $x_{15} = 10$ becomes the second basic variable (allocation), eliminating column 5 from further consideration.

The subsequent iterations proceed similarly, but you may want to test your understanding by verifying the remaining allocations given in Table 8.18. As with the other procedures in this (and other) section(s), you should find your OR Courseware useful for doing the calculations involved and illuminating the approach. (See the interactive routine for finding an initial BF solution.)

Comparison of Alternative Criteria for Step 1. Now let us compare these three criteria for selecting the next basic variable. The main virtue of the northwest corner rule is that it is quick and easy. However, because it pays no attention to unit costs c_{ij}, usually the solution obtained will be far from optimal. (Note in Table 8.16 that $x_{35} = 10$ even though $c_{35} = M$.) Expending a little more effort to find a good initial BF solution might greatly reduce the number of iterations then required by the transportation simplex method to reach an optimal solution (see Probs. 8.2-8 and 8.2-10). Finding such a solution is the objective of the other two criteria.

Vogel's approximation method has been a popular criterion for many years,[1] partially because it is relatively easy to implement by hand. Because the *difference* represents the minimum extra unit cost incurred by failing to make an allocation to the cell having the

[1]N. V. Reinfeld and W. R. Vogel, *Mathematical Programming,* Prentice-Hall, Englewood Cliffs, NJ, 1958.

smallest unit cost in that row or column, this criterion does take costs into account in an effective way.

Russell's approximation method provides another excellent criterion[1] that is still quick to implement on a computer (but not manually). Although is is unclear as to which is more effective *on average,* this criterion *frequently* does obtain a better solution than Vogel's. (For the example, Vogel's approximation method happened to find the optimal solution with $Z = 2{,}460$, whereas Russell's misses slightly with $Z = 2{,}570$.) For a large problem, it may be worthwhile to apply both criteria and then use the better solution to start the iterations of the transportation simplex method.

One distinct advantage of Russell's approximation method is that it is patterned directly after step 1 for the transportation simplex method (as you will see soon), which somewhat simplifies the overall computer code. In particular, the $\overline{u}_i$ and $\overline{v}_j$ values have been defined in such a way that the relative values of the $c_{ij} - \overline{u}_i - \overline{v}_j$ *estimate* the relative values of $c_{ij} - u_i - v_j$ that will be obtained when the transportation simplex method reaches an optimal solution.

We now shall use the initial BF solution obtained in Table 8.18 by Russell's approximation method to illustrate the remainder of the transportation simplex method. Thus, our *initial transportation simplex tableau* (before we solve for u_i and v_j) is shown in Table 8.19.

The next step is to check whether this initial solution is optimal by applying the *optimality test.*

[1]E. J. Russell, "Extension of Dantzig's Algorithm to Finding an Initial Near-Optimal Basis for the Transportation Problem," *Operations Research,* **17:** 187–191, 1969.

TABLE 8.19 Initial transportation simplex tableau (before we obtain $c_{ij} - u_i - v_j$) from Russell's approximation method

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16	16	13 (40)	22	17 (10)	50	
	2	14 (30)	14	13 (30)	19	15	60	
	3	19 (0)	19 (20)	20	23 (30)	*M*	50	
	4(*D*)	*M*	0	*M*	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j							

Optimality Test

Using the notation of Table 8.14, we can reduce the standard optimality test for the simplex method (see Sec. 4.3) to the following for the transportation problem:

> **Optimality test:** A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is nonbasic.[1]

Thus, the only work required by the optimality test is the derivation of the values of u_i and v_j for the current BF solution and then the calculation of these $c_{ij} - u_i - v_j$, as described below.

Since $c_{ij} - u_i - v_j$ is required to be zero if x_{ij} is a basic variable, u_i and v_j satisfy the set of equations

$$c_{ij} = u_i + v_j \qquad \text{for each } (i, j) \text{ such that } x_{ij} \text{ is basic.}$$

There are $m + n - 1$ basic variables, and so there are $m + n - 1$ of these equations. Since the number of unknowns (the u_i and v_j) is $m + n$, one of these variables can be assigned a value arbitrarily without violating the equations. The choice of this one variable and its value does not affect the value of any $c_{ij} - u_i - v_j$, even when x_{ij} is nonbasic, so the only (minor) difference it makes is in the ease of solving these equations. A convenient choice for this purpose is to select the u_i that has the *largest number of allocations in its row* (break any tie arbitrarily) and to assign to it the value zero. Because of the simple structure of these equations, it is then very simple to solve for the remaining variables algebraically.

To demonstrate, we give each equation that corresponds to a basic variable in our initial BF solution.

x_{31}:	$19 = u_3 + v_1$.	Set $u_3 = 0$, so $v_1 = 19$,
x_{32}:	$19 = u_3 + v_2$.	$v_2 = 19$,
x_{34}:	$23 = u_3 + v_4$.	$v_4 = 23$.
x_{21}:	$14 = u_2 + v_1$.	Know $v_1 = 19$, so $u_2 = -5$.
x_{23}:	$13 = u_2 + v_3$.	Know $u_2 = -5$, so $v_3 = 18$.
x_{13}:	$13 = u_1 + v_3$.	Know $v_3 = 18$, so $u_1 = -5$.
x_{15}:	$17 = u_1 + v_5$.	Know $u_1 = -5$, so $v_5 = 22$.
x_{45}:	$0 = u_4 + v_5$.	Know $v_5 = 22$, so $u_4 = -22$.

Setting $u_3 = 0$ (since row 3 of Table 8.19 has the largest number of allocations—3) and moving down the equations one at a time immediately give the derivation of values for the unknowns shown to the right of the equations. (Note that this derivation of the u_i and v_j values depends on which x_{ij} variables are *basic variables* in the current BF solution, so this derivation will need to be repeated each time a new BF solution is obtained.)

[1]The one exception is that two or more equivalent degenerate BF solutions (i.e., identical solutions having different degenerate basic variables equal to zero) can be optimal with only some of these basic solutions satisfying the optimality test. This exception is illustrated later in the example (see the identical solutions in the last two tableaux of Table 8.23, where only the latter solution satisfies the criterion for optimality).

Once you get the hang of it, you probably will find it even more convenient to solve these equations without writing them down by working directly on the transportation simplex tableau. Thus, in Table 8.19 you begin by writing in the value $u_3 = 0$ and then picking out the circled allocations (x_{31}, x_{32}, x_{34}) in that row. For each one you set $v_j = c_{3j}$ and then look for circled allocations (except in row 3) in these columns (x_{21}). Mentally calculate $u_2 = c_{21} - v_1$, pick out x_{23}, set $v_3 = c_{23} - u_2$, and so on until you have filled in all the values for u_i and v_j. (Try it.) Then calculate and fill in the value of $c_{ij} - u_i - v_j$ for each nonbasic variable x_{ij} (that is, for each cell without a circled allocation), and you will have the completed initial transportation simplex tableau shown in Table 8.20.

We are now in a position to apply the optimality test by checking the values of $c_{ij} - u_i - v_j$ given in Table 8.20. Because two of these values ($c_{25} - u_2 - v_5 = -2$ and $c_{44} - u_4 - v_4 = -1$) are negative, we conclude that the current BF solution is not optimal. Therefore, the transportation simplex method must next go to an iteration to find a better BF solution.

An Iteration

As with the full-fledged simplex method, an iteration for this streamlined version must determine an entering basic variable (step 1), a leaving basic variable (step 2), and then identify the resulting new BF solution (step 3).

Step 1. Since $c_{ij} - u_i - v_j$ represents the rate at which the objective function will change as the nonbasic variable x_{ij} is increased, the entering basic variable must have a *negative* $c_{ij} - u_i - v_j$ value to decrease the total cost Z. Thus, the candidates in Table 8.20 are x_{25} and x_{44}. To choose between the candidates, select the one having the larger (in absolute terms) negative value of $c_{ij} - u_i - v_j$ to be the entering basic variable, which is x_{25} in this case.

TABLE 8.20 Completed initial transportation simplex tableau

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 (40)	22 +4	17 (10)	50	−5
	2	14 (30)	14 0	13 (30)	19 +1	15 −2	60	−5
	3	19 (0)	19 (20)	20 +2	23 (30)	M $M-22$	50	0
	4(*D*)	M $M+3$	0 +3	M $M+4$	0 −1	0 (50)	50	−22
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j	19	19	18	23	22		

Step 2. Increasing the entering basic variable from zero sets off a *chain reaction* of compensating changes in other basic variables (allocations), in order to continue satisfying the supply and demand constraints. The first basic variable to be decreased to zero then becomes the leaving basic variable.

With x_{25} as the entering basic variable, the chain reaction in Table 8.20 is the relatively simple one summarized in Table 8.21. (We shall always indicate the entering basic variable by placing a boxed plus sign in the center of its cell while leaving the corresponding value of $c_{ij} - u_i - v_j$ in the lower right-hand corner of this cell.) Increasing x_{25} by some amount requires decreasing x_{15} by the same amount to restore the demand of 60 in column 5. This change then requires increasing x_{13} by this same amount to restore the supply of 50 in row 1. This change then requires decreasing x_{23} by this amount to restore the demand of 70 in column 3. This decrease in x_{23} successfully completes the chain reaction because it also restores the supply of 60 in row 2. (Equivalently, we could have started the chain reaction by restoring this supply in row 2 with the decrease in x_{23}, and then the chain reaction would continue with the increase in x_{13} and decrease in x_{15}.)

The net result is that cells (2, 5) and (1, 3) become **recipient cells,** each receiving its additional allocation from one of the **donor cells,** (1, 5) and (2, 3). (These cells are indicated in Table 8.21 by the plus and minus signs.) Note that cell (1, 5) had to be the donor cell for column 5 rather than cell (4, 5), because cell (4, 5) would have no recipient cell in row 4 to continue the chain reaction. [Similarly, if the chain reaction had been started in row 2 instead, cell (2, 1) could not be the donor cell for this row because the chain reaction could not then be completed successfully after necessarily choosing cell (3, 1) as the next recipient cell and either cell (3, 2) or (3, 4) as its donor cell.] Also note that, except for the entering basic variable, *all* recipient cells and donor cells in the chain reaction must correspond to *basic* variables in the current BF solution.

Each donor cell decreases its allocation by exactly the same amount as the entering basic variable (and other recipient cells) is increased. Therefore, the donor cell that starts with the smallest allocation—cell (1, 5) in this case (since $10 < 30$ in Table 8.21)—must reach a zero allocation first as the entering basic variable x_{25} is increased. Thus, x_{15} becomes the leaving basic variable.

TABLE 8.21 Part of initial transportation simplex tableau showing the chain reaction caused by increasing the entering basic variable x_{25}

		Destination				
			3	4	5	Supply
Source	1	…	13 (40)+	22 +4	17 (10)−	50
	2	…	13 (30)−	19 +1	15 [+] −2	60
		…	…	…	…	
Demand			70	30	60	

In general, there always is just *one* chain reaction (in either direction) that can be completed successfully to maintain feasibility when the entering basic variable is increased from zero. This chain reaction can be identified by selecting from the cells having a basic variable: first the donor cell in the *column* having the entering basic variable, then the recipient cell in the row having this donor cell, then the donor cell in the column having this recipient cell, and so on until the chain reaction yields a donor cell in the *row* having the entering basic variable. When a column or row has more than one additional basic variable cell, it may be necessary to trace them all further to see which one must be selected to be the donor or recipient cell. (All but this one eventually will reach a dead end in a row or column having no additional basic variable cell.) After the chain reaction is identified, the donor cell having the *smallest* allocation automatically provides the leaving basic variable. (In the case of a tie for the donor cell having the smallest allocation, any one can be chosen arbitrarily to provide the leaving basic variable.)

Step 3. The *new BF solution* is identified simply by adding the value of the leaving basic variable (before any change) to the allocation for each recipient cell and subtracting *this same amount* from the allocation for each donor cell. In Table 8.21 the value of the leaving basic variable x_{15} is 10, so the portion of the transportation simplex tableau in this table changes as shown in Table 8.22 for the new solution. (Since x_{15} is nonbasic in the new solution, its new allocation of zero is no longer shown in this new tableau.)

We can now highlight a useful interpretation of the $c_{ij} - u_i - v_j$ quantities derived during the optimality test. Because of the shift of 10 allocation units from the donor cells to the recipient cells (shown in Tables 8.21 and 8.22), the total cost changes by

$$\Delta Z = 10(15 - 17 + 13 - 13) = 10(-2) = 10(c_{25} - u_2 - v_5).$$

Thus, the effect of increasing the entering basic variable x_{25} from zero has been a cost change at the rate of -2 per unit increase in x_{25}. This is precisely what the value of $c_{25} - u_2 - v_5 = -2$ in Table 8.20 indicates would happen. In fact, another (but less efficient) way of deriving $c_{ij} - u_i - v_j$ for each nonbasic variable x_{ij} is to identify the chain reaction caused by increasing this variable from 0 to 1 and then to calculate the resulting cost change. This intuitive interpretation sometimes is useful for checking calculations during the optimality test.

TABLE 8.22 Part of second transportation simplex tableau showing the changes in the BF solution

		Destination				Supply
			3	4	5	
Source	1	…	13 (50)	22	17	50
	2	…	13 (20)	19	15 (10)	60
		…	…	…	…	
Demand			70	30	60	

Before completing the solution of the Metro Water District problem, we now summarize the rules for the transportation simplex method.

Summary of the Transportation Simplex Method.

Initialization: Construct an initial BF solution by the procedure outlined earlier in this section. Go to the optimality test.

Optimality test: Derive u_i and v_j by selecting the row having the largest number of allocations, setting its $u_i = 0$, and then solving the set of equations $c_{ij} = u_i + v_j$ for each (i, j) such that x_{ij} is basic. If $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is *nonbasic,* then the current solution is optimal, so stop. Otherwise, go to an iteration.

Iteration:

1. Determine the entering basic variable: Select the nonbasic variable x_{ij} having the *largest* (in absolute terms) *negative* value of $c_{ij} - u_i - v_j$.
2. Determine the leaving basic variable: Identify the chain reaction required to retain feasibility when the entering basic variable is increased. From the donor cells, select the basic variable having the *smallest* value.
3. Determine the new BF solution: Add the value of the leaving basic variable to the allocation for each recipient cell. Subtract this value from the allocation for each donor cell.

Continuing to apply this procedure to the Metro Water District problem yields the complete set of transportation simplex tableaux shown in Table 8.23. Since all the $c_{ij} - u_i - v_j$ values are nonnegative in the fourth tableau, the optimality test identifies the set of allocations in this tableau as being optimal, which concludes the algorithm.

It would be good practice for you to derive the values of u_i and v_j given in the second, third, and fourth tableaux. Try doing this by working directly on the tableaux. Also

TABLE 8.23 Complete set of transportation simplex tableaux for the Metro Water District problem

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 (40) +	22 +4	17 (10) −	50	−5
	2	14 (30)	14 0	13 (30) −	19 +1	15 + −2	60	−5
	3	19 (0)	19 (20)	20 +2	23 (30)	*M* *M* − 22	50	0
	4(*D*)	*M* *M* + 3	0 +3	*M* *M* + 4	0 −1	0 (50)	50	−22
Demand		30	20	70	30	60	*Z* = 2,570	
	v_j	19	19	18	23	22		

TABLE 8.23 (*Continued*)

Iteration 1		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 (50)	22 +4	17 +2	50	−5
	2	14 (30) −	14 0	13 (20)	19 +1	15 (10) +	60	−5
	3	19 (0) +	19 (20)	20 +2	23 (30) −	M M − 20	50	0
	4(*D*)	M M + 1	0 +1	M M + 2	0 + −3	0 (50) −	50	−20
Demand		30	20	70	30	60	Z = 2,550	
	v_j	19	19	18	23	20		

Iteration 2		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +5	16 +5	13 (50)	22 +7	17 +2	50	−8
	2	14 +3	14 +3	13 (20) −	19 +4	15 (40) +	60	−8
	3	19 (30)	19 (20)	20 + −1	23 (0) −	M M − 23	50	0
	4(*D*)	M M + 4	0 +4	M M + 2	0 (30) +	0 (20) −	50	−23
Demand		30	20	70	30	60	Z = 2,460	
	v_j	19	19	21	23	23		

Iteration 3		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +4	16 +4	13 (50)	22 +7	17 +2	50	−7
	2	14 +2	14 +2	13 (20)	19 +4	15 (40)	60	−7
	3	19 (30)	19 (20)	20 (0)	23 +1	M M − 22	50	0
	4(*D*)	M M + 3	0 +3	M M + 2	0 (30)	0 (20)	50	−22
Demand		30	20	70	30	60	Z = 2,460	
	v_j	19	19	20	22	22		

check out the chain reactions in the second and third tableaux, which are somewhat more complicated than the one you have seen in Table 8.21.

Note three special points that are illustrated by this example. First, the initial BF solution is *degenerate* because the basic variable $x_{31} = 0$. However, this degenerate basic variable causes no complication, because cell (3, 1) becomes a *recipient cell* in the second tableau, which increases x_{31} to a value greater than zero.

Second, another degenerate basic variable (x_{34}) arises in the third tableau because the basic variables for *two* donor cells in the second tableau, cells (2, 1) and (3, 4), *tie* for having the smallest value (30). (This tie is broken arbitrarily by selecting x_{21} as the leaving basic variable; if x_{34} had been selected instead, then x_{21} would have become the degenerate basic variable.) This degenerate basic variable does appear to create a complication subsequently, because cell (3, 4) becomes a *donor cell* in the third tableau but has nothing to donate! Fortunately, such an event actually gives no cause for concern. Since zero is the amount to be added to or subtracted from the allocations for the recipient and donor cells, these allocations do not change. However, the degenerate basic variable does become the leaving basic variable, so it is replaced by the entering basic variable as the circled allocation of zero in the fourth tableau. This change in the set of basic variables changes the values of u_i and v_j. Therefore, if any of the $c_{ij} - u_i - v_j$ had been negative in the fourth tableau, the algorithm would have gone on to make *real* changes in the allocations (whenever all donor cells have nondegenerate basic variables).

Third, because none of the $c_{ij} - u_i - v_j$ turned out to be negative in the fourth tableau, the equivalent set of allocations in the third tableau is optimal also. Thus, the algorithm executed one more iteration than was necessary. This extra iteration is a flaw that occasionally arises in both the transportation simplex method and the simplex method because of degeneracy, but it is not sufficiently serious to warrant any adjustments to these algorithms.

For another (smaller) example of the application of the transportation simplex method, refer to the demonstration provided for the transportation problem area in your OR Tutor. Also provided in your OR Courseware is an interactive routine for the transportation simplex method.

Now that you have studied the transportation simplex method, you are in a position to check for yourself how the algorithm actually provides a proof of the *integer solutions property* presented in Sec. 8.1. Problem 8.2-22 helps to guide you through the reasoning.

8.3 THE ASSIGNMENT PROBLEM

The **assignment problem** is a special type of linear programming problem where **assignees** are being assigned to perform **tasks.** For example, the assignees might be employees who need to be given work assignments. Assigning people to jobs is a common application of the assignment problem. However, the assignees need not be people. They also could be machines, or vehicles, or plants, or even time slots to be assigned tasks. The first example below involves machines being assigned to locations, so the tasks in this case simply involve holding a machine. A subsequent example involves plants being assigned products to be produced.

To fit the definition of an assignment problem, these kinds of applications need to be formulated in a way that satisfies the following assumptions.

1. The number of assignees and the number of tasks are the same. (This number is denoted by n.)
2. Each assignee is to be assigned to exactly *one* task.
3. Each task is to be performed by exactly *one* assignee.
4. There is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).
5. The objective is to determine how all n assignments should be made to minimize the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithms designed specifically for assignment problems.

The first three assumptions are fairly restrictive. Many potential applications do not quite satisfy these assumptions. However, it often is possible to reformulate the problem to make it fit. For example, *dummy assignees* or *dummy tasks* frequently can be used for this purpose. We illustrate these formulation techniques in the examples.

Prototype Example

The JOB SHOP COMPANY has purchased three new machines of different types. There are four available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centers that will have a heavy work flow to and from these machines. (There will be no work flow *between* the new machines.) Therefore, the objective is to assign the new machines to the available locations to minimize the total cost of materials handling. The estimated cost in dollars per hour of materials handling involving each of the machines is given in Table 8.24 for the respective locations. Location 2 is not considered suitable for machine 2, so no cost is given for this case.

To formulate this problem as an assignment problem, we must introduce a *dummy machine* for the extra location. Also, an extremely large cost M should be attached to the assignment of machine 2 to location 2 to prevent this assignment in the optimal solution. The resulting assignment problem *cost table* is shown in Table 8.25. This cost table contains all the necessary data for solving the problem. The optimal solution is to assign machine 1 to location 4, machine 2 to location 3, and machine 3 to location 1, for a total cost of \$29 per hour. The dummy machine is assigned to location 2, so this location is available for some future real machine.

We shall discuss how this solution is obtained after we formulate the mathematical model for the general assignment problem.

TABLE 8.24 Materials-handling cost data (\$) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

TABLE 8.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
	4(*D*)	0	0	0	0

The Assignment Problem Model and Solution Procedures

The mathematical model for the assignment problem uses the following decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Thus, each x_{ij} is a *binary variable* (it has value 0 or 1). As discussed at length in the chapter on integer programming (Chap. 12), binary variables are important in OR for representing *yes/no decisions.* In this case, the yes/no decision is: Should assignee i perform task j?

By letting Z denote the total cost, the assignment problem model is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j$$
$$(x_{ij} \text{ binary}, \qquad \text{for all } i \text{ and } j).$$

The first set of functional constraints specifies that each assignee is to perform exactly one task, whereas the second set requires each task to be performed by exactly one assignee. If we delete the parenthetical restriction that the x_{ij} be binary, the model clearly is a special type of linear programming problem and so can be readily solved. Fortunately, for reasons about to unfold, we *can* delete this restriction. (This deletion is the reason that the assignment problem appears in this chapter rather than in the integer programming chapter.)

Now compare this model (without the binary restriction) with the transportation problem model presented in the third subsection of Sec. 8.1 (including Table 8.6). Note how

similar their structures are. In fact, the assignment problem is just a special type of transportation problem where the *sources* now are *assignees* and the *destinations* now are *tasks* and where

Number of sources m = number of destinations n,

Every supply $s_i = 1$,

Every demand $d_j = 1$.

Now focus on the *integer solutions property* in the subsection on the transportation problem model. Because s_i and d_j are integers (= 1) now, this property implies that *every BF solution* (including an optimal one) is an *integer* solution for an assignment problem. The functional constraints of the assignment problem model prevent any variable from being greater than 1, and the nonnegativity constraints prevent values less than 0. Therefore, by deleting the binary restriction to enable us to solve an assignment problem as a linear programming problem, the resulting BF solutions obtained (including the final optimal solution) *automatically* will satisfy the binary restriction anyway.

Just as the transportation problem has a network representation (see Fig. 8.3), the assignment problem can be depicted in a very similar way, as shown in Fig. 8.5. The first column now lists the n assignees and the second column the n tasks. Each number in a

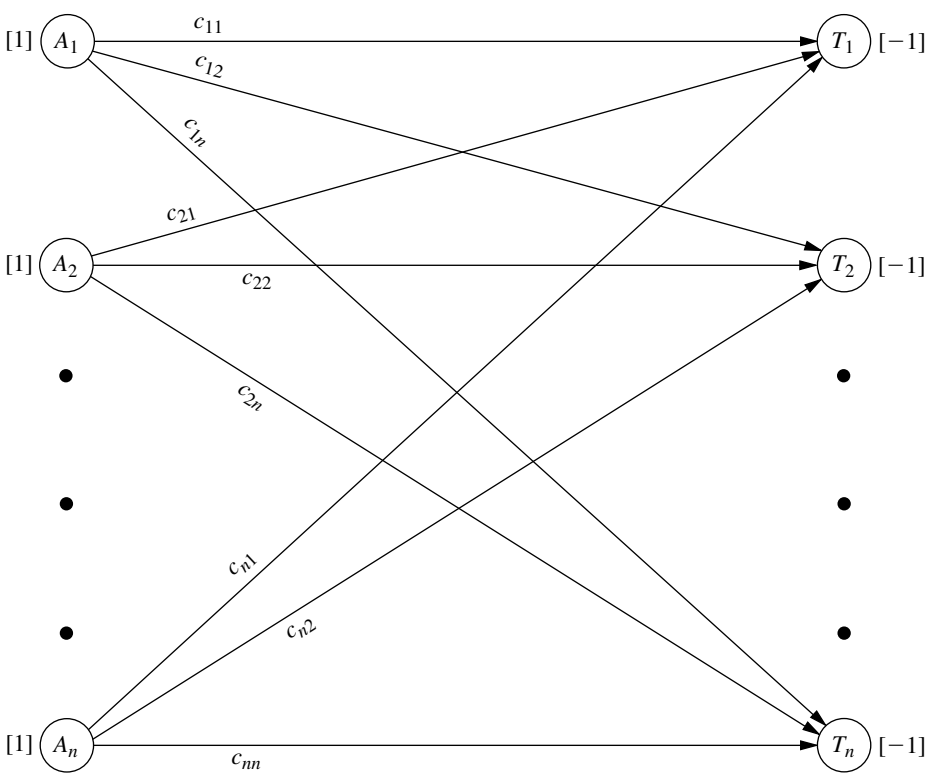


FIGURE 8.5
Network representation of the assignment problem.

square bracket indicates the number of assignees being provided at that location in the network, so the values are automatically 1 on the left, whereas the values of -1 on the right indicate that each task is using up one assignee.

For any particular assignment problem, practitioners normally do not bother writing out the full mathematical model. It is simpler to formulate the problem by filling out a cost table (e.g., Table 8.25), including identifying the assignees and tasks, since this table contains all the essential data in a far more compact form.

Alternative solution procedures are available for solving assignment problems. Problems that aren't much larger than the Job Shop Co. example can be solved very quickly by the general simplex method, so it may be convenient to simply use a basic software package (such as Excel and its Solver) that only employs this method. If this were done for the Job Shop Co. problem, it would not have been necessary to add the dummy machine to Table 8.25 to make it fit the assignment problem model. The constraints on the number of machines assigned to each location would be expressed instead as

$$\sum_{i=1}^{3} x_{ij} \leq 1 \qquad \text{for } j = 1, 2, 3, 4.$$

As shown in the Excel file for this chapter, a spreadsheet formulation for this example would be very similar to the formulation for a transportation problem displayed in Fig. 8.4 except now all the supplies and demands would be 1 and the demand constraints would be ≤ 1 instead of $= 1$.

However, large assignment problems can be solved much faster by using more specialized solution procedures, so we recommend using such a procedure instead of the general simplex method for big problems.

Because the assignment problem is a special type of transportation problem, one convenient and relatively fast way to solve any particular assignment problem is to apply the transportation simplex method described in Sec. 8.2. This approach requires converting the cost table to a parameter table for the equivalent transportation problem, as shown in Table 8.26*a*.

TABLE 8.26 Parameter table for the assignment problem formulated as a transportation problem, illustrated by the Job Shop Co. example

(a) General Case

		Cost per Unit Distributed				
		Destination				
		1	**2**	⋯	***n***	**Supply**
Source	1	c_{11}	c_{12}	⋯	c_{1n}	1
	2	c_{21}	c_{22}	⋯	c_{2n}	1
	⋮	⋯	⋯	⋯	⋯	⋮
	$m = n$	c_{n1}	c_{n2}	⋯	c_{nn}	1
Demand		1	1	⋯	1	

(b) Job Shop Co. Example

		Cost per Unit Distributed				
		Destination (Location)				
		1	**2**	**3**	**4**	**Supply**
Source (Machine)	1	13	16	12	11	1
	2	15	*M*	13	20	1
	3	5	7	10	6	1
	4(*D*)	0	0	0	0	1
Demand		1	1	1	1	

For example, Table 8.26*b* shows the parameter table for the Job Shop Co. problem that is obtained from the cost table of Table 8.25. When the transportation simplex method is applied to this transportation problem formulation, the resulting optimal solution has basic variables $x_{13} = 0$, $x_{14} = 1$, $x_{23} = 1$, $x_{31} = 1$, $x_{41} = 0$, $x_{42} = 1$, $x_{43} = 0$. (You are asked to verify this solution in Prob. 8.3-7.). The degenerate basic variables ($x_{ij} = 0$) and the assignment for the dummy machine ($x_{42} = 1$) do not mean anything for the original problem, so the real assignments are machine 1 to location 4, machine 2 to location 3, and machine 3 to location 1.

It is no coincidence that this optimal solution provided by the transportation simplex method has so many degenerate basic variables. For any assignment problem with n assignments to be made, the transportation problem formulation shown in Table 8.26*a* has $m = n$, that is, both the number of sources (m) and the number of destinations (n) in this formulation equal the number of assignments (n). Transportation problems in general have $m + n - 1$ basic variables (allocations), so every BF solution for this particular kind of transportation problem has $2n - 1$ basic variables, but exactly n of these x_{ij} equal 1 (corresponding to the n assignments being made). Therefore, since all the variables are binary variables, there always are $n - 1$ degenerate basic variables ($x_{ij} = 0$). As discussed at the end of Sec. 8.2, degenerate basic variables do not cause any major complication in the execution of the algorithm. However, they do frequently cause *wasted iterations,* where nothing changes (same allocations) except for the labeling of which allocations of zero correspond to degenerate basic variables rather than nonbasic variables. These wasted iterations are a major drawback to applying the transportation simplex method in this kind of situation, where there *always* are so many degenerate basic variables.

Another drawback of the transportation simplex method here is that it is purely a *general-purpose* algorithm for solving all transportation problems. Therefore, it does nothing to exploit the additional special structure in this special type of transportation problem ($m = n$, every $s_i = 1$, and every $d_j = 1$). Although we will not take the space to describe them,[1] specialized algorithms have been developed to fully streamline the procedure for solving just assignment problems. These algorithms operate directly on the cost table and do not bother with degenerate basic variables. When a computer code is available for one of these algorithms, it generally should be used in preference to the transportation simplex method, especially for really big problems. (The supplement to this chapter on the book's website, www.mhhe.com/hillier, describes one such algorithm.)

Example—Assigning Products to Plants

The BETTER PRODUCTS COMPANY has decided to initiate the production of four new products, using three plants that currently have excess production capacity. The products require a comparable production effort per unit, so the available production capacity of the plants is measured by the number of units of any product that can be produced per day, as given in the rightmost column of Table 8.27. The bottom row gives the required production rate per day to meet projected sales. Each plant can produce any of these prod-

[1]For an article comparing various algorithms for the assignment problem, see J. L. Kennington and Z. Wang, "An Empirical Analysis of the Dense Assignment Problem: Sequential and Parallel Implementations," *ORSA Journal on Computing,* **3:** 299–306, 1991.

TABLE 8.27 Data for the Better Products Co. problem

		Unit Cost ($) for Product				Capacity Available
		1	2	3	4	
Plant	1	41	27	28	24	75
	2	40	29	—	23	75
	3	37	30	27	21	45
Production rate		20	30	30	40	

ucts, *except* that Plant 2 *cannot* produce product 3. However, the variable costs per unit of each product differ from plant to plant, as shown in the main body of Table 8.27.

Management now needs to make a decision on how to split up the production of the products among plants. Two kinds of options are available.

Option 1: Permit *product splitting,* where the same product is produced in more than one plant.
Option 2: Prohibit *product splitting.*

This second option imposes a constraint that can only increase the cost of an optimal solution based on Table 8.27. On the other hand, the key advantage of Option 2 is that it eliminates some *hidden costs* associated with product splitting that are not reflected in Table 8.27, including extra setup, distribution, and administration costs. Therefore, management wants both options analyzed before a final decision is made. For Option 2, management further specifies that every plant should be assigned at least one of the products.

We will formulate and solve the model for each option in turn, where Option 1 leads to a transportation problem and Option 2 leads to an assignment problem.

Formulation of Option 1. With product splitting permitted, Table 8.27 can be converted directly to a parameter table for a transportation problem. The plants become the sources, and the products become the destinations (or vice versa), so the supplies are the available production capacities and the demands are the required production rates. Only two changes need to be made in Table 8.27. First, because Plant 2 cannot produce product 3, such an allocation is prevented by assigning to it a huge unit cost of M. Second, the total capacity ($75 + 75 + 45 = 195$) exceeds the total required production ($20 + 30 + 30 + 40 = 120$), so a dummy destination with a demand of 75 is needed to balance these two quantities. The resulting parameter table is shown in Table 8.28.

The optimal solution for this transportation problem has basic variables (allocations) $x_{12} = 30$, $x_{13} = 30$, $x_{15} = 15$, $x_{24} = 15$, $x_{25} = 60$, $x_{31} = 20$, and $x_{34} = 25$, so

Plant 1 produces all of products 2 and 3.
Plant 2 produces 37.5 percent of product 4.
Plant 3 produces 62.5 percent of product 4 and all of product 1.

The total cost is $Z = \$3{,}260$ per day.

Formulation of Option 2. Without product splitting, each product must be assigned to just one plant. Therefore, producing the products can be interpreted as the tasks for an assignment problem, where the plants are the assignees.

TABLE 8.28 Parameter table for the transportation problem formulation of Option 1 for the Better Products Co. problem

		Cost per Unit Distributed					
		Destination (Product)					
		1	2	3	4	5(*D*)	Supply
Source (Plant)	1	41	27	28	24	0	75
	2	40	29	*M*	23	0	75
	3	37	30	27	21	0	45
Demand		20	30	30	40	75	

Management has specified that every plant should be assigned at least one of the products. There are more products (four) than plants (three), so one of the plants will need to be assigned two products. Plant 3 has only enough excess capacity to produce one product (see Table 8.27), so *either* Plant 1 or Plant 2 will take the extra product.

To make this assignment of an extra product possible within an assignment problem formulation, Plants 1 and 2 each are split into two assignees, as shown in Table 8.29.

The number of assignees (now five) must equal the number of tasks (now four), so a *dummy task* (product) is introduced into Table 8.29 as 5(*D*). The role of this dummy task is to provide the fictional second product to either Plant 1 or Plant 2, whichever one receives only one real product. There is no cost for producing a fictional product so, as usual, the cost entries for the dummy task are zero. The one exception is the entry of *M* in the last row of Table 8.29. The reason for *M* here is that Plant 3 must be assigned a real product (a choice of product 1, 2, 3, or 4), so the Big *M* method is needed to prevent the assignment of the fictional product to Plant 3 instead. (As in Table 8.28, *M* also is used to prevent the infeasible assignment of product 3 to Plant 2.)

The remaining cost entries in Table 8.29 are *not* the unit costs shown in Table 8.27 or 8.28. Table 8.28 gives a transportation problem formulation (for Option 1), so unit costs are appropriate there, but now we are formulating an assignment problem (for Option 2). For an assignment problem, the cost c_{ij} is the *total* cost associated with assignee i performing task j. For Table 8.29, the *total cost* (per day) for Plant i to produce product j is the unit cost

TABLE 8.29 Cost table for the assignment problem formulation of Option 2 for the Better Products Co. problem

		Task (Product)				
		1	2	3	4	5(*D*)
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	*M*	920	0
	2*b*	800	870	*M*	920	0
	3	740	900	810	840	*M*

of production *times* the number of units produced (per day), where these two quantities for the multiplication are given separately in Table 8.27. For example, consider the assignment of Plant 1 to product 1. By using the corresponding unit cost in Table 8.28 ($41) and the corresponding demand (number of units produced per day) in Table 8.28 (20), we obtain

Cost of Plant 1 producing one unit of product 1	= $41
Required (daily) production of product 1	= 20 units
Total (daily) cost of assigning plant 1 to product 1	= 20 ($41)
	= $820

so 820 is entered into Table 8.29 for the cost of either Assignee 1a or 1b performing Task 1.

The optimal solution for this assignment problem is as follows:

Plant 1 produces products 2 and 3.
Plant 2 produces product 1.
Plant 3 produces product 4.

Here the dummy assignment is given to Plant 2. The total cost is $Z = \$3{,}290$ per day.

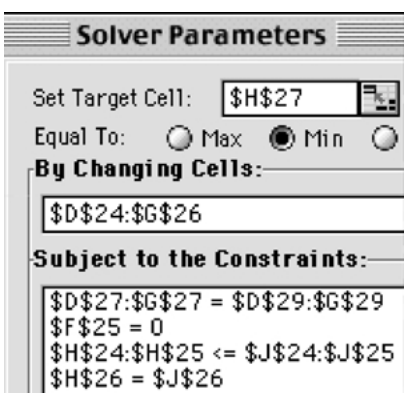
As usual, one way to obtain this optimal solution is to convert the cost table of Table 8.29 to a parameter table for the equivalent transportation problem (see Table 8.26) and then apply the transportation simplex method. Because of the identical rows in Table 8.29, this approach can be streamlined by combining the five assignees into three sources with supplies 2, 2, and 1, respectively. (See Prob. 8.3-6.) This streamlining also decreases by two the number of degenerate basic variables in every BF solution. Therefore, even though this streamlined formulation no longer fits the format presented in Table 8.26*a* for an assignment problem, it is a more efficient formulation for applying the transportation simplex method.

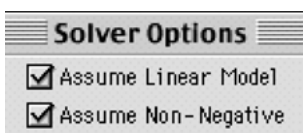
Figure 8.6 shows how Excel and its Solver can be used to obtain this optimal solution, which is displayed in the changing cells (D24:G26) of the spreadsheet. Since the general simplex method is being used, there is no need to fit this formulation into the format for either the assignment problem or transportation problem model. Therefore, the formulation does not bother to split Plants 1 and 2 into two assignees each, or to add a dummy task. Instead, Plants 1 and 2 are given a supply of 2 each, and then $\leq$ signs are entered into cells I24 and I25 as well as into the corresponding constraints in the Solver dialogue box. There also is no need to include the Big M method to prohibit assigning product 3 to Plant 2 in cell F25, since this dialogue box includes the constraint that F25 = 0. The target cell (H27) shows the total cost of $3,290 per day.

Now look back and compare this solution to the one obtained for Option 1, which included the splitting of product 4 between Plants 2 and 3. The allocations are somewhat different for the two solutions, but the total daily costs are virtually the same ($3,260 for Option 1 versus $3,290 for Option 2). However, there are hidden costs associated with product splitting (including the cost of extra setup, distribution, and administration) that are not included in the objective function for Option 1. As with any application of OR, the mathematical model used can provide only an approximate representation of the total problem, so management needs to consider factors that cannot be incorporated into the model before it makes a final decision. In this case, after evaluating the disadvantages of product splitting, management decided to adopt the Option 2 solution.

FIGURE 8.6
A spreadsheet formulation of Option 2 for the Better Products Co. problem as a variant of an assignment problem, where rows 12 to 18 show the cost table and the changing cells (D24:G26) display the optimal production plan obtained by the Solver.

	A	B	C	D	E	F	G	H	I	J
1	Better Products Co. Production-Planning Problem (Option 2)									
2										
3				Unit Cost						
4				Product						
5				1	2	3	4			
6			1	$41	$27	$28	$24			
7		Plant	2	$40	$29	-	$23			
8			3	$37	$30	$27	$21			
9		Required Production		20	30	30	40			
10										
11										
12				Cost ($/day)						
13				Task (Product)						
14				1	2	3	4	Supply		
15		Assignee	1	$820	$810	$840	$960	2		
16		(Plant)	2	$800	$870	-	$920	2		
17			3	$740	$900	$810	$840	1		
18		Demand		1	1	1	1			
19										
20										
21				Assignments						
22				Task (Product)						
23				1	2	3	4	Totals		Supply
24		Assignee	1	0	1	1	0	2	≤	2
25		(Plant)	2	1	0	0	0	1	≤	2
26			3	0	0	0	1	1	=	1
27		Totals		1	1	1	1	$ 3,290	=	Total Cost
28				=	=	=	=			
29		Demand		1	1	1	1			





	H
24	=SUM(D24:G24)
25	=SUM(D25:G25)
26	=SUM(D26:G26)
27	=SUMPRODUCT(D15:G17,D24:G26)

	D	E	F	G
15	=D6*D$9	=E6*E$9	=F6*F$9	=G6*G$9
16	=D7*D$9	=E7*E$9	-	=G7*G$9
17	=D8*D$9	=E8*E$9	=F8*F$9	=G8*G$9
27	=SUM(D24:D26)	=SUM(E24:E26)	=SUM(F24:F26)	=SUM(G24:G26)

8.4 CONCLUSIONS

The linear programming model encompasses a wide variety of specific types of problems. The general simplex method is a powerful algorithm that can solve surprisingly large versions of any of these problems. However, some of these problem types have such simple formulations that they can be solved much more efficiently by *streamlined* algorithms that exploit their *special structure.* These streamlined algorithms can cut down tremendously on the computer time required for large problems, and they sometimes make it computationally feasible to solve huge problems. This is particularly true for the two types of linear programming problems studied in this chapter, namely, the transportation problem and the assignment problem. Both types have a number of common applications, so it is important to recognize them when they arise and to use the best available algorithms. These special-purpose algorithms are included in some linear programming software packages.

We shall reexamine the special structure of the transportation and assignment problems in Sec. 9.6. There we shall see that these problems are special cases of an important class of linear programming problems known as the *minimum cost flow problem.* This problem has the interpretation of minimizing the cost for the flow of goods through a network. A streamlined version of the simplex method called the *network simplex method* (described in Sec. 9.7) is widely used for solving this type of problem, including its various special cases.

A supplementary chapter (Chap. 23) on the book's website, www.mhhe.com/hillier, describes various additional special types of linear programming problems. One of these, called the *transshipment problem,* is a generalization of the transportation problem which allows shipments from any source to any destination to first go through intermediate transfer points. Since the transshipment problem also is a special case of the minimum cost flow problem, we will describe it further in Sec. 9.6.

Much research continues to be devoted to developing streamlined algorithms for special types of linear programming problems, including some not discussed here. At the same time, there is widespread interest in applying linear programming to optimize the operation of complicated large-scale systems. The resulting formulations usually have special structures that can be exploited. Being able to recognize and exploit special structures is an important factor in the successful application of linear programming.

SELECTED REFERENCES

1. Bazaraa, M. S., J. J. Jarvis, and H. D. Sherali: *Linear Programming and Network Flows,* 2d ed., Wiley, New York, 1990, chap. 10.

2. Dantzig, G. B., and M. N. Thapa: *Linear Programming 1: Introduction,* Springer, New York, 1997, chap. 8.

3. Geoffrion, A. M.: "Elements of Large-Scale Mathematical Programming," *Management Science,* **16:** 652–691, 1970.

4. Hillier, F. S., M. S. Hillier, and G. J. Lieberman: *Introduction to Management Science: A Modeling and Case Studies Approach with Spreadsheets,* Irwin/McGraw-Hill, Burr Ridge, IL, 2000, chap. 6.

5. Murthy, D. N. P., N. W. Page, and E. Y. Rodin: *Mathematical Modeling: A Tool for Problem Solving in Engineering, Physical, Biological and Social Sciences,* Pergamon Press, Oxford, England, 1990.

LEARNING AIDS FOR THIS CHAPTER IN YOUR OR COURSEWARE

A Demonstration Example in OR Tutor:

The Transportation Problem

Interactive Routines:

Enter or Revise a Transportation Problem
Find Initial Basic Feasible Solution—for Interactive Method
Solve Interactively by the Transportation Simplex Method

An Excel Add-in:

Premium Solver

"Ch. 8—Transp. & Assignment" Files for Solving the Examples:

Excel File
LINGO/LINDO File
MPL/CPLEX File

Supplement to this Chapter:

An Algorithm for the Assignment Problem (appears on the book's website, www.mhhe.com/hillier)

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example listed above may be helpful.
I: We suggest that you use the corresponding interactive routines listed above (the printout records your work).
C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

8.1-1. The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given to the right:

		Distance			
		Distribution Center			
		1	2	3	4
Plant	1	800 miles	1,300 miles	400 miles	700 miles
	2	1,100 miles	1,400 miles	600 miles	1,000 miles
	3	600 miles	1,200 miles	800 miles	900 miles

The freight cost for each shipment is $100 plus 50 cents per mile.

How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
(b) Draw the network representation of this problem.
C **(c)** Obtain an optimal solution.

8.1-2.* Tom would like 3 pints of home brew today and an additional 4 pints of home brew tomorrow. Dick is willing to sell a maximum of 5 pints total at a price of $3.00 per pint today and $2.70 per pint tomorrow. Harry is willing to sell a maximum of 4 pints total at a price of $2.90 per pint today and $2.80 per pint tomorrow.

Tom wishes to know what his purchases should be to minimize his cost while satisfying his thirst requirements.

(a) Formulate a *linear programming* model for this problem, and construct the initial simplex tableau (see Chaps. 3 and 4).

(b) Formulate this problem as a *transportation problem* by constructing the appropriate parameter table.

C **(c)** Obtain an optimal solution.

8.1-3. The Versatech Corporation has decided to produce three new products. Five branch plants now have excess product capacity. The unit manufacturing cost of the first product would be $31, $29, $32, $28, and $29 in Plants 1, 2, 3, 4, and 5, respectively. The unit manufacturing cost of the second product would be $45, $41, $46, $42, and $43 in Plants 1, 2, 3, 4, and 5, respectively. The unit manufacturing cost of the third product would be $38, $35, and $40 in Plants 1, 2, and 3, respectively, whereas Plants 4 and 5 do not have the capability for producing this product. Sales forecasts indicate that 600, 1,000, and 800 units of products 1, 2, and 3, respectively, should be produced per day. Plants 1, 2, 3, 4, and 5 have the capacity to produce 400, 600, 400, 600, and 1,000 units daily, respectively, regardless of the product or combination of products involved. Assume that any plant having the capability and capacity to produce them can produce any combination of the products in any quantity.

Management wishes to know how to allocate the new products to the plants to minimize total manufacturing cost.

(a) Formulate this problem as a *transportation problem* by constructing the appropriate parameter table.

C **(b)** Obtain an optimal solution.

8.1-4. Suppose that England, France, and Spain produce all the wheat, barley, and oats in the world. The world demand for wheat requires 125 million acres of land devoted to wheat production. Similarly, 60 million acres of land are required for barley and 75 million acres of land for oats. The total amount of land available for these purposes in England, France, and Spain is 70 million acres, 110 million acres, and 80 million acres, respectively. The number of hours of labor needed in England, France, and Spain to produce an acre of wheat is 18, 13, and 16, respectively. The number of hours of labor needed in England, France, and Spain to produce an acre of barley is 15, 12, and 12, respectively. The number of hours of labor needed in England, France, and Spain to produce an acre of oats is 12, 10, and 16, respectively. The labor cost per hour in producing wheat is $9.00, $7.20, and $9.90 in England, France, and Spain, respectively. The labor cost per hour in producing barley is $8.10, $9.00, and $8.40 in England, France, and Spain, respectively. The labor cost per hour in producing oats is $6.90, $7.50, and $6.30 in England, France, and Spain, respectively. The problem is to allocate land use in each country so as to meet the world food requirement and minimize the total labor cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

(b) Draw the network representation of this problem.

C **(c)** Obtain an optimal solution.

8.1-5. Reconsider the P & T Co. problem presented in Sec. 8.1. You now learn that one or more of the shipping costs per truckload given in Table 8.2 may change slightly before shipments begin.

Use the Excel Solver to generate the Sensitivity Report for this problem. Use this report to determine the allowable range to stay optimal for each of the unit costs. What do these allowable ranges tell P & T management?

8.1-6. The Onenote Co. produces a single product at three plants for four customers. The three plants will produce 60, 80, and 40 units, respectively, during the next time period. The firm has made a commitment to sell 40 units to customer 1, 60 units to customer 2, and at least 20 units to customer 3. Both customers 3 and 4 also want to buy as many of the remaining units as possible. The net profit associated with shipping a unit from plant i for sale to customer j is given by the following table:

		Customer			
		1	**2**	**3**	**4**
Plant	1	$800	$700	$500	$200
	2	$500	$200	$100	$300
	3	$600	$400	$300	$500

Management wishes to know how many units to sell to customers 3 and 4 and how many units to ship from each of the plants to each of the customers to maximize profit.

(a) Formulate this problem as a transportation problem where the objective function is to be maximized by constructing the appropriate parameter table that gives unit profits.

(b) Now formulate this transportation problem with the usual objective of minimizing total cost by converting the parameter table from part (*a*) into one that gives unit costs instead of unit profits.

(c) Display the formulation in part (*a*) on an Excel spreadsheet.

C **(d)** Use this information and the Excel Solver to obtain an optimal solution.

C **(e)** Repeat parts (*c*) and (*d*) for the formulation in part (*b*). Compare the optimal solutions for the two formulations.

8.1-7. The Move-It Company has two plants producing forklift trucks that then are shipped to three distribution centers. The production costs are the same at the two plants, and the cost of shipping for each truck is shown for each combination of plant and distribution center:

		Distribution Center		
		1	2	3
Plant	A	$800	$700	$400
	B	$600	$800	$500

A total of 60 forklift trucks are produced and shipped per week. Each plant can produce and ship any amount up to a maximum of 50 trucks per week, so there is considerable flexibility on how to divide the total production between the two plants so as to reduce shipping costs. However, each distribution center must receive exactly 20 trucks per week.

Management's objective is to determine how many forklift trucks should be produced at each plant, and then what the overall shipping pattern should be to minimize total shipping cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

(b) Display the transportation problem on an Excel spreadsheet.

C **(c)** Use the Excel Solver to obtain an optimal solution.

8.1-8. Redo Prob. 8.1-7 when any distribution center may receive any quantity between 10 and 30 forklift trucks per week in order to further reduce total shipping cost, provided only that the total shipped to all three distribution centers must still equal 60 trucks per week.

8.1-9. The Build-Em-Fast Company has agreed to supply its best customer with three widgets during *each* of the next 3 weeks, even though producing them will require some overtime work. The relevant production data are as follows:

Week	Maximum Production, Regular Time	Maximum Production, Overtime	Production Cost per Unit, Regular Time
1	2	2	$300
2	3	2	$500
3	1	2	$400

The cost per unit produced with overtime for each week is $100 more than for regular time. The cost of storage is $50 per unit for each week it is stored. There is already an inventory of two widgets on hand currently, but the company does not want to retain any widgets in inventory after the 3 weeks.

Management wants to know how many units should be produced in each week to minimize the total cost of meeting the delivery schedule.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

C **(b)** Obtain an optimal solution.

8.1-10. The MJK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced on Regular Time (RT) and on Overtime (O). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced on Regular Time, (3) the cost (in thousands of dollars) per unit produced on Overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for Product 1 on the left and the number for Product 2 on the right.

	Maximum Combined Production		Product 1/Product 2			
				Unit Cost of Production ($1,000's)		Unit Cost of Storage
Month	RT	OT	Sales	RT	OT	($1,000's)
1	10	3	5/3	15/16	18/20	1/2
2	8	2	3/5	17/15	20/18	2/1
3	10	3	4/4	19/17	22/22	

The production manager wants a schedule developed for the number of units of each of the two products to be produced on Regular Time and (if Regular Time production capacity is used up) on Overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

C **(b)** Obtain an optimal solution.

8.2-1. Consider the transportation problem having the following parameter table:

		Destination			
		1	**2**	**3**	**Supply**
Source	1	6	3	5	4
	2	4	*M*	7	3
	3	3	4	3	2
Demand		4	2	3	

(a) Use Vogel's approximation method manually (don't use the interactive routine in your OR Courseware) to select the first basic variable for an initial BF solution.

(b) Use Russell's approximation method manually to select the first basic variable for an initial BF solution.

(c) Use the northwest corner rule manually to construct a complete initial BF solution.

D,I **8.2-2.*** Consider the transportation problem having the following parameter table:

		Destination					
		1	**2**	**3**	**4**	**5**	**Supply**
Source	1	2	4	6	5	7	4
	2	7	6	3	*M*	4	6
	3	8	7	5	2	5	6
	4	0	0	0	0	0	4
Demand		4	4	2	5	5	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.

(a) Northwest corner rule.

(b) Vogel's approximation method.

(c) Russell's approximation method.

D,I **8.2-3.** Consider the transportation problem having the following parameter table:

		Destination						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
Source	1	13	10	22	29	18	0	5
	2	14	13	16	21	*M*	0	6
	3	3	0	*M*	11	6	0	7
	4	18	9	19	23	11	0	4
	5	30	24	34	36	28	0	3
Demand		3	5	4	5	6	2	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.

(a) Northwest corner rule.

(b) Vogel's approximation method.

(c) Russell's approximation method.

8.2-4. Consider the transportation problem having the following parameter table:

		Destination				
		1	**2**	**3**	**4**	**Supply**
Source	1	7	4	1	4	1
	2	4	6	7	2	1
	3	8	5	4	6	1
	4	6	7	6	3	1
Demand		1	1	1	1	

(a) Notice that this problem has three special characteristics: (1) number of sources = number of destinations, (2) each supply = 1, and (3) each demand = 1. Transportation problems with these characteristics are of a special type called the assignment problem (as described in Sec. 8.3). Use the integer solutions property to explain why this type of transportation problem can be interpreted as assigning sources to destinations as a one-to-one basis.

(b) How many basic variables are there in every BF solution? How many of these are degenerate basic variables (= 0)?

D,I **(c)** Use the northwest corner rule to obtain an initial BF solution.

I **(d)** Construct an initial BF solution by applying the general procedure for the initialization step of the transportation simplex method. However, rather than using one of the three criteria for step 1 presented in Sec. 8.2, use the minimum cost criterion given next for selecting the next basic variable. (With the corresponding interactive routine in your OR Courseware, choose the *Northwest Corner Rule,* since this choice actually allows the use of any criterion.)

Minimum cost criterion: From among the rows and columns still under consideration, select the variable x_{ij} having the smallest unit cost c_{ij} to be the next basic variable. (Ties may be broken arbitrarily.)

D,I **(e)** Starting with the initial BF solution from part (*c*), interactively apply the transportation simplex method to obtain an optimal solution.

8.2-5. Consider the prototype example for the transportation problem (the P & T Co. problem) presented at the beginning of Sec. 8.1. Verify that the solution given there actually is optimal by applying just the *optimality test* portion of the transportation simplex method to this solution.

8.2-6. Consider the transportation problem formulation of Option 1 for the Better Products Co. problem presented in Table 8.28. Verify that the optimal solution given in Sec. 8.3 actually is optimal by applying just the *optimality test* portion of the transportation simplex method to this solution.

8.2-7. Consider the transportation problem having the following parameter table:

		Destination					
		1	**2**	**3**	**4**	**5**	**Supply**
Source	1	8	6	3	7	5	20
	2	5	*M*	8	4	7	30
	3	6	3	9	6	8	30
	4(*D*)	0	0	0	0	0	20
Demand		25	25	20	10	20	

After several iterations of the transportation simplex method, a BF solution is obtained that has the following basic variables: $x_{13} = 20$, $x_{21} = 25$, $x_{24} = 5$, $x_{32} = 25$, $x_{34} = 5$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 20$. Continue the transportation simplex method for *two more* iterations by hand. After two iterations, state whether the solution is optimal and, if so, why.

D,I **8.2-8.*** Consider the transportation problem having the following parameter table:

		Destination				
		1	**2**	**3**	**4**	**Supply**
Source	1	3	7	6	4	5
	2	2	4	3	2	2
	3	4	3	8	5	3
Demand		3	3	2	2	

Use each of the following criteria to obtain an initial BF solution. In each case, interactively apply the transportation simplex method, starting with this initial solution, to obtain an optimal solution. Compare the resulting number of iterations for the transportation simplex method.

(a) Northwest corner rule.
(b) Vogel's approximation method.
(c) Russell's approximation method.

D,I **8.2-9.** The Cost-Less Corp. supplies its four retail outlets from its four plants. The shipping cost per shipment from each plant to each retail outlet is given below.

		Unit Shipping Cost Retail Outlet			
		1	**2**	**3**	**4**
Plant	1	$500	$600	$400	$200
	2	$200	$900	$100	$300
	3	$300	$400	$200	$100
	4	$200	$100	$300	$200

Plants 1, 2, 3, and 4 make 10, 20, 20, and 10 shipments per month, respectively. Retail outlets 1, 2, 3, and 4 need to receive 20, 10, 10, and 20 shipments per month, respectively.

The distribution manager, Randy Smith, now wants to determine the best plan for how many shipments to send from each plant to the respective retail outlets each month. Randy's objective is to minimize the total shipping cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
(b) Use the northwest corner rule to construct an initial BF solution.
(c) Starting with the initial basic solution from part (*b*), interactively apply the transportation simplex method to obtain an optimal solution.

8.2-10. The Energetic Company needs to make plans for the energy systems for a new building.

The energy needs in the building fall into three categories: (1) electricity, (2) heating water, and (3) heating space in the building. The daily requirements for these three categories (all measured in the same units) are

Electricity	20 units
Water heating	10 units
Space heating	30 units.

The three possible sources of energy to meet these needs are electricity, natural gas, and a solar heating unit that can be installed on the roof. The size of the roof limits the largest possible solar heater

to 30 units, but there is no limit to the electricity and natural gas available. Electricity needs can be met only by purchasing electricity (at a cost of $50 per unit). Both other energy needs can be met by any source or combination of sources. The unit costs are

	Electricity	Natural Gas	Solar Heater
Water heating	$90	$60	$30
Space heating	$80	$50	$40

The objective is to minimize the total cost of meeting the energy needs.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

D,I **(b)** Use the northwest corner rule to obtain an initial BF solution for this problem.

D,I **(c)** Starting with the initial BF solution from part (*b*), interactively apply the transportation simplex method to obtain an optimal solution.

D,I **(d)** Use Vogel's approximation method to obtain an initial BF solution for this problem.

D,I **(e)** Starting with the initial BF solution from part (*d*), interactively apply the transportation simplex method to obtain an optimal solution.

I **(f)** Use Russell's approximation method to obtain an initial BF solution for this problem.

D,I **(g)** Starting with the initial BF solution obtained from part (*f*), interactively apply the transportation simplex method to obtain an optimal solution. Compare the number of iterations required by the transportation simplex method here and in parts (*c*) and (*e*).

D,I **8.2-11.*** Interactively apply the transportation simplex method to solve the Northern Airplane Co. production scheduling problem as it is formulated in Table 8.9.

D,I **8.2-12.*** Reconsider Prob. 8.1-1.

(a) Use the northwest corner rule to obtain an initial BF solution.

(b) Starting with the initial BF solution from part (*a*), interactively apply the transportation simplex method to obtain an optimal solution.

D,I **8.2-13.** Reconsider Prob. 8.1-2*b.* Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

D,I **8.2-14.** Reconsider Prob. 8.1-3. Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

D,I **8.2-15.** Reconsider Prob. 8.1-4. Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

D,I **8.2-16.** Reconsider Prob. 8.1-6. Starting with Russell's approximation method, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

8.2-17. Reconsider the transportation problem formulated in Prob. 8.1-7*a.*

D,I **(a)** Use each of the three criteria presented in Sec. 8.2 to obtain an initial BF solution, and time how long you spend for each one. Compare both these times and the values of the objective function for these solutions.

C **(b)** Obtain an optimal solution for this problem. For each of the three initial BF solutions obtained in part (*a*), calculate the percentage by which its objective function value exceeds the optimal one.

D,I **(c)** For each of the three initial BF solutions obtained in part (*a*), interactively apply the transportation simplex method to obtain (and verify) an optimal solution. Time how long you spend in each of the three cases. Compare both these times and the number of iterations needed to reach an optimal solution.

8.2-18. Follow the instructions of Prob. 8.2-17 for the transportation problem formulated in Prob. 8.1-8*a.*

8.2-19. Consider the transportation problem having the following parameter table:

		Destination		
		1	2	Supply
Source	1	8	5	4
	2	6	4	2
Demand		3	3	

(a) Using your choice of a criterion from Sec. 8.2 for obtaining the initial BF solution, solve this problem manually by the transportation simplex method. (Keep track of your time.)

(b) Reformulate this problem as a general linear programming problem, and then solve it manually by the *simplex method.* Keep track of how long this takes you, and contrast it with the computation time for part (*a*).

8.2-20. Consider the Northern Airplane Co. production scheduling problem presented in Sec. 8.1 (see Table 8.7). Formulate this problem as a general linear programming problem by letting the

decision variables be x_j = number of jet engines to be produced in month j ($j = 1, 2, 3, 4$). Construct the initial simplex tableau for this formulation, and then contrast the size (number of rows and columns) of this tableau and the corresponding tableaux used to solve the transportation problem formulation of the problem (see Table 8.9).

8.2-21. Consider the general linear programming formulation of the transportation problem (see Table 8.6). Verify the claim in Sec. 8.2 that the set of $(m + n)$ functional constraint equations (m supply constraints and n demand constraints) has one *redundant* equation; i.e., any one equation can be reproduced from a linear combination of the other $(m + n - 1)$ equations.

8.2-22. When you deal with a transportation problem where the supply and demand quantities have *integer* values, explain why the steps of the transportation simplex method guarantee that all the basic variables (allocations) in the BF solutions obtained must have integer values. Begin with why this occurs with the initialization step when the general procedure for constructing an *initial* BF solution is used (regardless of the criterion for selecting the next basic variable). Then given a *current* BF solution that is integer, next explain why Step 3 of an iteration must obtain a new BF solution that also is integer. Finally, explain how the initialization step can be used to construct *any* initial BF solution, so the transportation simplex method actually gives a proof of the integer solutions property presented in Sec. 8.1.

8.2-23. A contractor, Susan Meyer, has to haul gravel to three building sites. She can purchase as much as 18 tons at a gravel pit in the north of the city and 14 tons at one in the south. She needs 10, 5, and 10 tons at sites 1, 2, and 3, respectively. The purchase price per ton at each gravel pit and the hauling cost per ton are given in the table below.

	Hauling Cost per Ton at Site			
Pit	**1**	**2**	**3**	**Price per Ton**
North	\$30	\$60	\$50	\$100
South	\$60	\$30	\$40	\$120

Susan wishes to determine how much to haul from each pit to each site to minimize the total cost for purchasing and hauling gravel.

(a) Formulate a linear programming model for this problem. Using the Big M method, construct the initial simplex tableau ready to apply the simplex method (but do not actually solve).

(b) Now formulate this problem as a transportation problem by constructing the appropriate parameter table. Compare the size of this table (and the corresponding transportation simplex tableau) used by the transportation simplex method with the size of the simplex tableaux from part (*a*) that would be needed by the simplex method.

D **(c)** Susan Meyer notices that she can supply sites 1 and 2 completely from the north pit and site 3 completely from the south pit. Use the optimality test (but no iterations) of the transportation simplex method to check whether the corresponding BF solution is optimal.

D,I **(d)** Starting with the northwest corner rule, interactively apply the transportation simplex method to solve the problem as formulated in part (*b*).

(e) As usual, let c_{ij} denote the unit cost associated with source i and destination j as given in the parameter table constructed in part (*b*). For the optimal solution obtained in part (*d*), suppose that the value of c_{ij} for each basic variable x_{ij} is fixed at the value given in the parameter table, but that the value of c_{ij} for each nonbasic variable x_{ij} possibly can be altered through bargaining because the site manager wants to pick up the business. Use sensitivity analysis to determine the *allowable range to stay optimal* for each of the latter c_{ij}, and explain how this information is useful to the contractor.

C **8.2-24.** Consider the transportation problem formulation and solution of the Metro Water District problem presented in Secs. 8.1 and 8.2 (see Tables 8.12 and 8.23).

The numbers given in the parameter table are only estimates that may be somewhat inaccurate, so management now wishes to do some what-if analysis. Use the Excel Solver to generate the Sensitivity Report. Then use this report to address the following questions. (In each case, assume that the indicated change is the only change in the model.)

(a) Would the optimal solution in Table 8.23 remain optimal if the cost per acre foot of shipping Calorie River water to San Go were actually \$200 rather than \$230?

(b) Would this solution remain optimal if the cost per acre foot of shipping Sacron River water to Los Devils were actually \$160 rather than \$130?

(c) Must this solution remain optimal if the costs considered in parts (*a*) and (*b*) were simultaneously changed from their original values to \$215 and \$145, respectively?

(d) Suppose that the supply from the Sacron River and the demand at Hollyglass are decreased simultaneously by the same amount. Must the shadow prices for evaluating these changes remain valid if the decrease were 0.5 million acre feet?

8.2-25. Without generating the Sensitivity Report, adapt the sensitivity analysis procedure presented in Secs. 6.6 and 6.7 to conduct the sensitivity analysis specified in the four parts of Prob. 8.2-24.

8.3-1. Consider the assignment problem having the following cost table.

		Task			
		1	**2**	**3**	**4**
Assignee	A	8	6	5	7
	B	6	5	3	4
	C	7	8	4	6
	D	6	7	5	6

(a) Draw the network representation of this assignment problem.
(b) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
(c) Display this formulation on an Excel spreadsheet.
C **(d)** Use the Excel Solver to obtain an optimal solution.

8.3-2. Four cargo ships will be used for shipping goods from one port to four other ports (labeled 1, 2, 3, 4). Any ship can be used for making any one of these four trips. However, because of differences in the ships and cargoes, the total cost of loading, transporting, and unloading the goods for the different ship-port combinations varies considerably, as shown in the following table:

		Port			
		1	**2**	**3**	**4**
Ship	1	$500	$400	$600	$700
	2	$600	$600	$700	$500
	3	$700	$500	$700	$600
	4	$500	$400	$600	$600

The objective is to assign the four ships to four different ports in such a way as to minimize the total cost for all four shipments.
(a) Describe how this problem fits into the general format for the assignment problem.
C **(b)** Obtain an optimal solution.
(c) Reformulate this problem as an equivalent transportation problem by constructing the appropriate parameter table.
D,I **(d)** Use the northwest corner rule to obtain an initial BF solution for the problem as formulated in part (*c*).
D,I **(e)** Starting with the initial BF solution from part (*d*), interactively apply the transportation simplex method to obtain an optimal set of assignments for the original problem.
D,I **(f)** Are there other optimal solutions in addition to the one obtained in part (*e*)? If so, use the transportation simplex method to identify them.

8.3-3. Reconsider Prob. 8.1-3. Suppose that the sales forecasts have been revised downward to 240, 400, and 320 units per day of products 1, 2, and 3, respectively, and that each plant now has the capacity to produce all that is required of any one product. Therefore, management has decided that each new product should be assigned to only one plant and that no plant should be assigned more than one product (so that three plants are each to be assigned one product, and two plants are to be assigned none). The objective is to make these assignments so as to minimize the *total* cost of producing these amounts of the three products.
(a) Formulate this problem as an assignment problem by constructing the appropriate cost table.
C **(b)** Obtain an optimal solution.
(c) Reformulate this assignment problem as an equivalent transportation problem by constructing the appropriate parameter table.
D,I **(d)** Starting with Vogel's approximation method, interactively apply the transportation simplex method to solve the problem as formulated in part (*c*).

8.3-4.* The coach of an age group swim team needs to assign swimmers to a 200-yard medley relay team to send to the Junior Olympics. Since most of his best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five fastest swimmers and the best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

Stroke	**Carl**	**Chris**	**David**	**Tony**	**Ken**
Backstroke	37.7	32.9	33.8	37.0	35.4
Breaststroke	43.4	33.1	42.2	34.7	41.8
Butterfly	33.3	28.5	38.9	30.4	33.6
Freestyle	29.2	26.4	29.6	28.5	31.1

The coach wishes to determine how to assign four swimmers to the four different strokes to minimize the sum of the corresponding best times.
(a) Formulate this problem as an assignment problem.
C **(b)** Obtain an optimal solution.

8.3-5. Reconsider Prob. 8.2-23. Now suppose that trucks (and their drivers) need to be hired to do the hauling, where each truck can only be used to haul gravel from a single pit to a single site. Each truck can haul 5 tons, and the cost per truck is five times the hauling cost per ton given earlier. Only full trucks would be used to supply each site.
(a) Formulate this problem as an assignment problem by constructing the appropriate cost table, including identifying the assignees and tasks.

C **(b)** Obtain an optimal solution.
(c) Reformulate this assignment problem as an equivalent transportation problem with two sources and three destinations by constructing the appropriate parameter table.
C **(d)** Obtain an optimal solution for the problem as formulated in part (*c*).

8.3-6. Consider the assignment problem formulation of Option 2 for the Better Products Co. problem presented in Table 8.29.
(a) Reformulate this problem as an equivalent transportation problem with three sources and five destinations by constructing the appropriate parameter table.
(b) Convert the optimal solution given in Sec. 8.3 for this assignment problem into a complete BF solution (including degenerate basic variables) for the transportation problem formulated in part (*a*). Specifically, apply the "General Procedure for Constructing an Initial BF Solution" given in Sec. 8.2. For each iteration of the procedure, rather than using any of the three alternative criteria presented for step 1, select the next basic variable to correspond to the next assignment of a plant to a product given in the optimal solution. When only one row or only one column remains under consideration, use step 4 to select the remaining basic variables.
(c) Verify that the optimal solution given in Sec. 8.3 for this assignment problem actually is optimal by applying just the optimality test portion of the transportation simplex method to the complete BF solution obtained in part (*b*).
(d) Now reformulate this assignment problem as an equivalent transportation problem with five sources and five destinations by constructing the appropriate parameter table. Compare this transportation problem with the one formulated in part (*a*).
(e) Repeat part (*b*) for the problem as formulated in part (*d*). Compare the BF solution obtained with the one from part (*b*).

D,I **8.3-7.** Starting with Vogel's approximation method, interactively apply the transportation simplex method to solve the Job Shop Co. assignment problem as formulated in Table 8.26*b*. (As stated in Sec. 8.3, the resulting optimal solution has $x_{14} = 1$, $x_{23} = 1$, $x_{31} = 1$, $x_{42} = 1$, and all other $x_{ij} = 0$.)

8.3-8. Reconsider Prob. 8.1-7. Now assume that distribution centers 1, 2, and 3 must receive exactly 10, 20, and 30 units per week, respectively. For administrative convenience, management has decided that each distribution center will be supplied totally by a single plant, so that one plant will supply one distribution center and the other plant will supply the other two distribution centers. The choice of these assignments of plants to distribution centers is to be made solely on the basis of minimizing total shipping cost.
(a) Formulate this problem as an assignment problem by constructing the appropriate cost table, including identifying the corresponding assignees and tasks.
C **(b)** Obtain an optimal solution.
(c) Reformulate this assignment problem as an equivalent transportation problem (with four sources) by constructing the appropriate parameter table.
C **(d)** Solve the problem as formulated in part (*c*).
(e) Repeat part (*c*) with just two sources.
C **(f)** Solve the problem as formulated in part (*e*).

8.3-9. Consider the assignment problem having the following cost table.

		Job		
		1	**2**	**3**
Person	A	5	7	4
	B	3	6	5
	C	2	3	4

The optimal solution is A-3, B-1, C-2, with $Z = 10$.
C **(a)** Use the computer to verify this optimal solution.
(b) Reformulate this problem as an equivalent transportation problem by constructing the appropriate parameter table.
C **(c)** Obtain an optimal solution for the transportation problem formulated in part (*b*).
(d) Why does the optimal BF solution obtained in part (*c*) include some (degenerate) basic variables that are not part of the optimal solution for the assignment problem?
(e) Now consider the *nonbasic* variables in the optimal BF solution obtained in part (*c*). For each nonbasic variable x_{ij} and the corresponding cost c_{ij}, adapt the sensitivity analysis procedure for general linear programming (see Case 2*a* in Sec. 6.7) to determine the *allowable range to stay optimal for* c_{ij}.

8.3-10. Consider the linear programming model for the general assignment problem given in Sec. 8.3. Construct the table of constraint coefficients for this model. Compare this table with the one for the general transportation problem (Table 8.6). In what ways does the general assignment problem have more special structure than the general transportation problem?

CASE 8.1 SHIPPING WOOD TO MARKET

Alabama Atlantic is a lumber company that has three sources of wood and five markets to be supplied. The annual availability of wood at sources 1, 2, and 3 is 15, 20, and 15 million board feet, respectively. The amount that can be sold annually at markets 1, 2, 3, 4, and 5 is 11, 12, 9, 10, and 8 million board feet, respectively.

In the past the company has shipped the wood by train. However, because shipping costs have been increasing, the alternative of using ships to make some of the deliveries is being investigated. This alternative would require the company to invest in some ships. Except for these investment costs, the shipping costs in thousands of dollars per million board feet by rail and by water (when feasible) would be the following for each route:

	Unit Cost by Rail ($1,000's) Market					Unit Cost by Ship ($1,000's) Market				
Source	**1**	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
1	61	72	45	55	66	31	38	24	—	35
2	69	78	60	49	56	36	43	28	24	31
3	59	66	63	61	47	—	33	36	32	26

The capital investment (in thousands of dollars) in ships required for each million board feet to be transported annually by ship along each route is given as follows:

	Investment for Ships ($1,000's) Market				
Source	**1**	**2**	**3**	**4**	**5**
1	275	303	238	—	285
2	293	318	270	250	265
3	—	283	275	268	240

Considering the expected useful life of the ships and the time value of money, the equivalent uniform annual cost of these investments is one-tenth the amount given in the table. The objective is to determine the overall shipping plan that minimizes the total equivalent uniform annual cost (including shipping costs).

You are the head of the OR team that has been assigned the task of determining this shipping plan for each of the following three options.

Option 1: Continue shipping exclusively by rail.
Option 2: Switch to shipping exclusively by water (except where only rail is feasible).
Option 3: Ship by either rail or water, depending on which is less expensive for the particular route.

Present your results for each option. Compare.

Finally, consider the fact that these results are based on *current* shipping and investment costs, so that the decision on the option to adopt now should take into account management's projection of how these costs are likely to change in the future. For each option, describe a scenario of future cost changes that would justify adopting that option now.

CASE 8.2 PROJECT PICKINGS

Tazer, a pharmaceutical manufacturing company, entered the pharmaceutical market 12 years ago with the introduction of six new drugs. Five of the six drugs were simply permutations of existing drugs and therefore did not sell very heavily. The sixth drug, however, addressed hypertension and was a huge success. Since Tazer had a patent on the hypertension drug, it experienced no competition, and profits from the hypertension drug alone kept Tazer in business.

During the past 12 years, Tazer continued a moderate amount of research and development, but it never stumbled upon a drug as successful as the hypertension drug. One reason is that the company never had the motivation to invest heavily in innovative research and development. The company was riding the profit wave generated by its hypertension drug and did not feel the need to commit significant resources to finding new drug breakthroughs.

Now Tazer is beginning to fear the pressure of competition. The patent for the hypertension drug expires in 5 years,[1] and Tazer knows that once the patent expires, generic drug manufacturing companies will swarm into the market like vultures. Historical trends show that generic drugs decreased sales of branded drugs by 75 percent.

Tazer is therefore looking to invest significant amounts of money in research and development this year to begin the search for a new breakthrough drug that will offer the company the same success as the hypertension drug. Tazer believes that if the company begins extensive research and development now, the probability of finding a successful drug shortly after the expiration of the hypertension patent will be high.

As head of research and development at Tazer, you are responsible for choosing potential projects and assigning project directors to lead each of the projects. After researching the needs of the market, analyzing the shortcomings of current drugs, and interviewing numerous scientists concerning the promising areas of medical research, you have decided that your department will pursue five separate projects, which are listed below:

Project Up	Develop an antidepressant that does not cause serious mood swings.
Project Stable	Develop a drug that addresses manic-depression.
Project Choice	Develop a less intrusive birth control method for women.
Project Hope	Develop a vaccine to prevent HIV infection.
Project Release	Develop a more effective drug to lower blood pressure.

[1]In general, patents protect inventions for 17 years. In 1995, GATT legislation extending the protection given by new pharmaceutical patents to 20 years became effective. The patent for Tazer's hypertension drug was issued prior to the GATT legislation, however. Thus, the patent only protects the drug for 17 years.

For each of the five projects, you are only able to specify the medical ailment the research should address, since you do not know what compounds will exist and be effective without research.

You also have five senior scientists to lead the five projects. You know that scientists are very temperamental people and will work well only if they are challenged and motivated by the project. To ensure that the senior scientists are assigned to projects they find motivating, you have established a bidding system for the projects. You have given each of the five scientists 1000 bid points. They assign bids to each project, giving a higher number of bid points to projects they most prefer to lead. The following table provides the bids from the five individual senior scientists for the five individual projects:

Project	Dr. Kvaal	Dr. Zuner	Dr. Tsai	Dr. Mickey	Dr. Rollins
Project Up	100	0	100	267	100
Project Stable	400	200	100	153	33
Project Choice	200	800	100	99	33
Project Hope	200	0	100	451	34
Project Release	100	0	600	30	800

You decide to evaluate a variety of scenarios you think are likely.

(a) Given the bids, you need to assign one senior scientist to each of the five projects to maximize the preferences of the scientists. What are the assignments?

(b) Dr. Rollins is being courted by Harvard Medical School to accept a teaching position. You are fighting desperately to keep her at Tazer, but the prestige of Harvard may lure her away. If this were to happen, the company would give up the project with the least enthusiasm. Which project would not be done?

(c) You do not want to sacrifice any project, since researching only four projects decreases the probability of finding a breakthrough new drug. You decide that either Dr. Zuner or Dr. Mickey could lead two projects. Under these new conditions with just four senior scientists, which scientists will lead which projects to maximize preferences?

(d) After Dr. Zuner was informed that she and Dr. Mickey are being considered for two projects, she decided to change her bids. The following table shows Dr. Zuner's new bids for each of the projects:

Project Up	20
Project Stable	450
Project Choice	451
Project Hope	39
Project Release	40

Under these new conditions with just four scientists, which scientists will lead which projects to maximize preferences?

(e) Do you support the assignment found in part (*d*)? Why or why not?

(f) Now you again consider all five scientists. You decide, however, that several scientists cannot lead certain projects. In particular, Dr. Mickey does not have experience with research on the immune system, so he cannot lead Project Hope. His family also has a history of manic-depression, and you feel that he would be too personally involved in Project Stable

to serve as an effective project leader. Dr. Mickey therefore cannot lead Project Stable. Dr. Kvaal also does not have experience with research on the immune systems and cannot lead Project Hope. In addition, Dr. Kvaal cannot lead Project Release because he does not have experience with research on the cardiovascular system. Finally, Dr. Rollins cannot lead Project Up because her family has a history of depression and you feel she would be too personally involved in the project to serve as an effective leader. Because Dr. Mickey and Dr. Kvaal cannot lead two of the five projects, they each have only 600 bid points. Dr. Rollins has only 800 bid points because she cannot lead one of the five projects. The following table provides the new bids of Dr. Mickey, Dr. Kvaal, and Dr. Rollins:

Project	Dr. Mickey	Dr. Kvaal	Dr. Rollins.
Project Up	300	86	Can't lead
Project Stable	Can't lead	343	50
Project Choice	125	171	50
Project Hope	Can't lead	Can't lead	100
Project Release	175	Can't lead	600

Which scientists should lead which projects to maximize preferences?

(g) You decide that Project Hope and Project Release are too complex to be led by only one scientist. Therefore, each of these projects will be assigned two scientists as project leaders. You decide to hire two more scientists in order to staff all projects: Dr. Arriaga and Dr. Santos. Because of religious reasons, the two doctors both do not want to lead Project Choice. The following table lists all projects, scientists, and their bids.

	Kvaal	Zuner	Tsai	Mickey	Rollins	Arriaga	Santos
Up	86	0	100	300	Can't lead	250	111
Stable	343	200	100	Can't lead	50	250	1
Choice	171	800	100	125	50	Can't lead	Can't lead
Hope	Can't lead	0	100	Can't lead	100	250	333
Release	Can't lead	0	600	175	600	250	555

Which scientists should lead which projects to maximize preferences?

(h) Do you think it is wise to base your decision in part (*g*) only on an optimal solution for an assignment problem?